**Mathematics 552** **Algebra II** **Spring 2013**

**Week 7** Wrapup of semisimple algebras, Brauer Group and Group representations
Jacobson II: 4.5-4.7, 5.1-5.3

1. The statements below will prove: Let $K$ be a field and let $D_1, D_2$ be division algebras over $K$ of dimensions $n_1, n_2$ (the centers may be larger than $K$). If $n1, n2$ are relatively prime, then $D_1 \otimes_K D_2$ is a division algebra. (a generalization of part of Jacobson II 4.6.9 from problem set 6)

   a) Show that for any simple K-algebras $A_1, A_2$ there is a simple algebra $B$ and a surjective $K$-algebra homomorphism $A_1 \otimes A_2 \to B$. (Hint: take the quotient by maximal two-sided ideal of $A_1 \otimes A_2$).

   b) For division algebras of finite dimension over $K$ consider an algebra $B$ constructed in a) as a vector space under $D_1 \otimes 1$. Show that $dim_K B = dim_{D_1} B dim_K D_1 \leq dim_K D_1 dim_K D_2$.

   c) If $dim_K D_1$ is relatively prime to $dim_K D_2$ show that $dim_K B = dim_K D_1 dim_K D_2$ so that $D_1 \otimes_K D_2$ is a simple algebra.

   d) Under the assumption of c), let $L$ be an irreducible ideal of the simple ring $A = D_1 \otimes_K D_2 = M_r(E)$, where the division algebra $E = (End_A(L))^{opp}$. Then $A = L^r$ as an $A$-module. $L$ is a module over $D1 \simeq D_1 \otimes 1$ and $D_2 \simeq 1 \otimes D_2$. Compute the dimension of $A$ as vector space over $D_1$ and $D_2$ and show that $r$ divides the dimensions of $D_1, D_2$ as vectors spaces over $K$. Conclude that if these dimensions are relatively prime, then $r = 1$ , establishing the statement at the beginning of this problem.

2. Let $G$ be a finite group and consider representations of $G$ on finite dimensional complex vector spaces.

   a) Let $V, W$ be irreducible representations of $G$. Show that the decomposition of $V^* \otimes W$ into irreducible representations contains exactly 1 copy of the trivial representation of $G$ if $V, W$ are isomorphic, and 0 otherwise (see previously assigned Jacobson II 5.3.7).

   b) Show that the element $t = \sum_{g \in G} g$ in $\mathbf{C}[G]$ is in the center of the group algebra, hence by Schur's lemma acts by a scalar on any irreducible representation $V$. Show that if the subspace $W = tV \subset V$ is nonzero, then $V$ is the trivial representation. Show that $\chi_V(t) = 0$ if $V$ is not the trivial representation, and $\chi_V(t) = |G|$ if $V$ is the trivial representation.

c) Show using a),b) that if $V, W$ are irreducible representations then $\chi_{V*\otimes W}(t)$ is 0 or $|G|$ according as $V, W$ are not isomorphic or are isomorphic. Deduce that the characters of irreducible representations of $G$ form an orthonormal family of functions with respect to the pairing $<\phi, \psi>= (1/|G|)\sum_{g\in G}\phi(g^{-1})\psi(g)$.

d) Show that if a representation $V$ of $G$ decomposes as a sum of irreducibles $V = \oplus m_i V_i$ with $V_i, V_j$ not isomorphic when $i \neq j$, then $<\chi_V, \chi_V>= \sum m_i^2$. Show that $V$ is irreducible if and only if $<\chi_V, \chi_V>= 1$.

e) Show that if $W = \oplus m_i V_i$ is the decomposition of $W$ into distinct irreducibles, then $<\chi_W, \chi_{V_i}>= m_i$. This, together with d) allows one to check from characters if a representation is irreducible, and if so how many times it appears in another representation.

3. The complex representations of $S_4$, the symmetric group on four letters.

   a) Find the conjugacy classes in $S_4$, find all normal subgroups, and determine how many irreducible complex representations $S_4$ has.

   b) Show that the group $G$ of rotations preserving a cube permutes the 4 diagonals joining opposite corners of the cube and that this gives an injective group homomorphism from $G$ to $S_4$. Show that $G$ acts transitively on the 8 vertices of the cube and compute the stabilizer in $G$ of a vertex. Show that the group $G$ of rotations is isomorphic to $S_4$

   c) Show that the group $G$ of rotations of the cube maps onto the group of permutations of the lines joining centers of opposite faces and that $S_3$ is a quotient of $G$. Use this to find some irreducible representations of $S_4$ and to determine the dimension of the remaining irreducible representations.

   d) Show that the natural representation of $G$ as linear maps of $R^3$ is irreducible

   e) Let $W$ be the representation of $G$ on the vector space of complex functions on the faces of the cube. Compute the character of $W$ and express $W$ as a sum of irreducible representations.

4. Let $G$ be a subgroup of the group $GL(n, \mathbf{C})$ of invertible $n \times n$ complex matrices. Suppose that $\sum_{g\in G} trace(g) = 0$. Show that $\sum_{g\in G} g = 0$.