Mathematics 552 **Algebra II** **Spring 2013**

Week 5 Semisimple rings, the Artin-Wedderburn Theorem and Group Representations
Jacobson II: 4.5, 5.1, 5.2

1. Let V be an infinite dimensional vector space over a division ring D with countable basis $e_1, e_2, \ldots$. Let $R = End_D(V)$ be the ring of F-linear maps of V to itself, with product the composition of maps.

   a) Show that the set $I \subset R$ of all linear maps with finite dimensional image is a two-sided ideal in R.

   b) Show that the only two sided ideals of R are $0, R, I$. Show that I is a maximal two-sided ideal in R and that the ring R/I is a simple ring.

   c) Show that the set I_j consisting of all endomorphisms vanishing on e_n when 2^j divides n is a left ideal of R. Show that the left ideals $I + I_j$ are distinct for distinct j and form an increasing chain of left ideals and that R/I is a simple ring which is not semisimple.

2. Let K be a field, let G be a group and let $K[G]$ be the group ring of G.

   a) Show that if kernel of the map $K[G] \to K$ which sends $\sum a_g g \in K[G]$ to $\sum a_g$ has a complement that it is a one dimensional K-vector space which is an eigenspace for the linear transformation multiplication by g on $K[G]$.

   b Show that when the characteristic of K is a prime number p and $K[G]$ is a semisimple ring then G has no elements of order p.

   c) Show that the converse of Maschke's theorem holds: The group ring $K[G]$ of a finite group G is semisimple if and only if the order of G is not divisible by the characteristic of K.

3. Let R be a ring.

   a) Show that R is semisimple if and only if every left R-module is projective.

   b) Show that R is a division algebra if and only if every left R-module is free.

4. Show that if R is a semisimple ring and M is a finitely generated left R-module, then $End_R(M)$ is a semisimple ring.