Mathematics 552 **Algebra II** **Spring 2013**

Text Jacobson, *Basic Algebra I and II*

References Lang, *Algebra*
Hungerford, *Algebra* , GTM 73
MacLane, *Categories for the Working Mathematician*
van der Waerden, *Modern Algebra*
Kostrikin and Shafarevitch, *Algebra I*

Week 2 The Dehn invariant, the Cauchy-Binet formula
Jacobson I:7.1-7.2, Jacobson II: 1.1-1.3 , 3.1, 3.7-3.9

1. Let A and B be abelian groups.
   a) Let m be a positive integer and let $a \in A$ be divisible by m in the sense that $a = ma'$ for some $a' \in A$. Show that $a \otimes b = a' \otimes mb$ in $A \otimes B$.
   b) Let m and a be as in part a). Let $b \in B$ be an element of an abelian group B such that $mb = 0$ in B. Show that the element $a \otimes b$ is the identity element of $A \otimes B$.
   c) Let A be an abelian group such that A is divisible in the sense that $A = mA$ for all positive integers m. Suppose that B is a torsion abelian group (that is, all elements of B have finite order). Show that $A \otimes B = 0$. Find an example of a nontrivial abelian group A such that $A \otimes A = 0$.
2. Show that in $\mathbf{R} \otimes \mathbf{R}/\pi\mathbf{Z}$ we have that for any nonzero integer m that $l \otimes \alpha/m = l/m \otimes \alpha$. Show that given an element $z \in \mathbf{R} \otimes \mathbf{R}/\pi\mathbf{Z}$ there is a finite set S of real numbers (perhaps empty) such that $z = \sum_{s \in S} l_s \otimes s$ such that the union of S and π is a linearly independent set over the rational field. Show that z is zero in the tensor product if and only if all $l_s = 0$. Show that $\mathbf{R} \otimes \mathbf{R}/\pi\mathbf{Z}$ is a torsion free abelian group.
3. Use the formula which computes the $k \times k$ minors of the product of a matrix A of size $m \times n$ and a matrix B of size $n \times p$ in terms of $k \times k$ minors of A and B in the special case $m = p = 2$ to prove Lagrange's identity: given elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ of a commutative ring R then

$$(\sum_{l=1}^{n} a_l^2)(\sum_{l=1}^{n} b_l^2) - (\sum_{l=1}^{n} a_l b_l)^2$$

   is a sum of squares in R. (Note this generalizes the Cauchy-Schwarz inequality for the case R the real numbers).