Mathematics 552 **Algebra II** **Spring 2013**

The exterior algebra of K-modules and the Cauchy-Binet formula

Let K be a commutative ring, and let M be a K-module. Recall that the tensor algebra $T(M) = \oplus_{i=0} M^{\otimes i}$ is an associative K-algebra and that if M is free of finite rank m with basis $v_1, \ldots, v_m$ then $T(M)$ is a graded free K module with basis for $M^{\otimes k}$ given by the collection of $v_{i_1} \otimes v_{i_2} \otimes \cdots v_{i_k}$ for $1 \leq i_t \leq m$. Hence $M^{\otimes k}$ is a free K-module of rank m^k.

We checked in class that the tensor algebra was functorial, in that when $f : M \to N$ is a K-module homomorphism, then the induced map $f^{\otimes i} : M^{\otimes i} \to N^{\otimes i}$ gives a homomorphism of K-algebras $T(f) : T(M) \to T(N)$ and that this gives a functor from K-modules to K-algebras.

The ideal $I \subset T(M)$ generated by elements of the form $x \otimes x$ for $x \in M$ enables us to construct an algebra $\Lambda(M) = T(M)/I$. Since $(v+w)\otimes(v+w) = v\otimes v+v\otimes w+w\otimes v+w\otimes w$ we have that (modulo I) $v \otimes w = -w \otimes v$. The image of $M^{\otimes i}$ in $\Lambda(M)$ is denoted $\Lambda^i(M)$, the i-th exterior power of M. The product in $\Lambda(M)$ derived from the product on $T(M)$ is usually called the wedge product : $v \wedge w$ is the image of $v \otimes w$ in $T(M)/I$.

Since making tensor algebra of modules is functorial, so is making the exterior algebra. In particular given a homomorphism of K-modules $f : M \to N$ there is a homomorphism $\Lambda(f) : \Lambda(M) \to \Lambda(N)$ and homomorphisms $\Lambda^k(f) : \Lambda^k(M) \to \Lambda^k(N)$.

If M is free of finite rank m with basis $v_1, \ldots, v_m$ then $\Lambda(M)$ is a graded free K module with basis for $\Lambda^k(M)$ given by $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$ with $i_1 < i_2 < \ldots < i_k$ (recall that $v_i \wedge v_j = -v_j \wedge v_i$ and that $v_i \wedge v_i = 0$). In this case $\Lambda^k(M)$ is a free K-module of rank $\binom{n}{k}$, the number of k-element subsets of an n-element set. Thus $\Lambda^k(M) = 0$ if $k > m$, and $\Lambda^m(M) = Kv_1 \wedge \cdots \wedge v_m$ has rank 1 and is isomorphic to K. In this situation any K-module endomorphism $f : M \to M$ gives a K-module map $\Lambda^m(f) : K \to K$ which is determined by the image of 1, which we will denote by $det(f)$. For example, if m=2, and $f(v_1) = av_1 + bv_2, f(v_2) = cv_1 + dv_2$ then $\Lambda^2(f)(v_1 \wedge v_2) = (av_1 + bv_2) \wedge (cv_1 + dv_2) = (ad - bc)v_1 \wedge v_2$ so the determinant of f is $det(f) = ad - bc$.

By functoriality, if $f : M \to M, g : M \to M$ are K-endomorphisms of M then $\Lambda^m(g \circ f) = \Lambda^m(g) \circ \Lambda^m(f)$ so that $\det(g \circ f) = \det(f) \det(g)$.

When M is free over K of rank m with basis v_i , and N is free of rank n with basis w_j each K homomorphism $f : M \to N$ determines an $m \times n$ matrix $A = \{a_{ij}\}$ via $f(v_i) = \sum_j a_{ij} w_j$. The module $\Lambda^k(M)$ has a basis $\{v_S\}$ indexed by k-element subsets $S \subset \{1, \ldots, m\}$ and $\Lambda^k(N)$ has basis $\{w_T\}$ indexed by k-element subsets $T \subset \{1, \ldots, n\}$. To compute the matrix of $\Lambda^k(f)$ is to compute the coefficient a_{ST} of W_T in $\Lambda^k(f)(v_S)$. Given S, T as above we can compute this by considering the submodule V spanned by $v_i : i \in S$ of M and quotient module W of N spanned by $w_j : j \in T$. The sequence $V \to M \to N \to W$ where the middle arrow is the homomorphism f gives a homomorphism $V \to W$ associated to the $k \times k$ submatrix of elements of the matrix of f lying in rows indexed by S and columns indexed by T. Taking the k-th exterior power gives a sequence $\Lambda^k(V) \to \Lambda^k(M) \to \Lambda^k(N) \to \Lambda^k(W)$ showing that a_{ST} is the 1×1 matrix associated to the homomorphism from the first to the last module in the sequence. This is the determinant of the $k \times k$ submatrix of A formed by elements in rows indexed by S and columns indexed by T, which we will call the ST minor A_{ST} of A.

If P is free of rank p, the composition of f with a K endomorphism $g : N \to P$ associated to an $n \times p$ matrix $B = \{b_{kl}\}$ is easily computed to be associated with the usual matrix product AB. Taking k-th exterior powers we can compute the matrices associated to $\Lambda^k(f), \Lambda^k(g), \Lambda^k(g \circ f) = \Lambda^k(g) \circ \Lambda^k(f)$ as in the previous paragraph. These matrices involve $k \times k$ minors of A, B and AB.

The Cauchy-Binet formula is an explicit realization in the special case of finite rank free modules over K of the fact that taking exterior powers is a functor . Let S, T, U be k-element subsets of $\{1, \ldots, m\}, \{1, \ldots, n\}, \{1, \ldots, p\}$ respectively. Then the SU minor of AB is the SU entry of the matrix of $\Lambda^k(g \circ f)$ which from the above is a product of matrices involving minors of $\Lambda^k(g)$ and $\Lambda^k(f)$.

$$\mathrm{Cauchy-BinetFormula:} \qquad (AB)_{SU} = \sum_T A_{ST} B_{TU}$$

where the sum runs over all k-element subsets $T \subset \{1, \ldots, n\}$.

The left hand side is an entry in the matrix of $\Lambda^k(g \circ f)$. Since this is the transformation $\Lambda^k(g) \circ \Lambda^k(f)$ the right hand side is obtained as the result of multiplying the associated matrices.

The case $k = 1$ is the usual formula for multiplying matrices (1×1 minors are just matrix entries). The case $m = n = p = k$ is the multiplicativity of the determinant of $n \times n$ matrices.

The first interesting case is $k = 2$, which is relevant to problem 3. of problem set 2.