Mathematics 551 **Algebra** **Fall 2006**

Brief notes on projective and injective modules

Let R be a ring. We are interested in the category $\mathcal{R}-\mathcal{MOD}$ of left R–modules. Recall that this is an abelian category. Given an R–module B there are two functors of interest from $\mathcal{R}-\mathcal{MOD}$ to the category $\mathcal{AB}$ of abelian groups.

$$h_B(M) = Hom_{\mathcal{R}-\mathcal{MOD}}(B, M)$$

$$h'_B(M) = Hom_{\mathcal{R}-\mathcal{MOD}}(M, B)$$

The first is a covariant functor: given a module homomorphism $f : M \to N$ we obtain $h_B(f) : h_B(M) \to h_B(N)$ by $h_B(f)(\phi) = f \circ \phi$ for all module homomorphisms $\phi \in Hom_{\mathcal{R}-\mathcal{MOD}}(B, M)$. Similarly h'_B is a contravariant functor: given ψ an element of $Hom_{\mathcal{R}-\mathcal{MOD}}(N, B)$ and $f : M \to N$ the homomorphism $h'_B(f) = \psi \circ f$ is in $Hom_{\mathcal{R}-\mathcal{MOD}}(M, B)$.

Recall that in an abelian category the image of a morphism $f : A \to B$ is defined to be ker coker$f$ where the kernel (resp cokernel) of a morphism is the equalizer (reps. coequalizer) of the morphism with the zero morphism. A sequence of morphisms $\cdots A \to B \to C \to \cdots$ is exact at B if the image of $A \to B$ equals the kernel of $B \to C$. This is equivalent to the exactness of the short sequence $0 \to \text{image}(A \to B) \to B \to \text{coker}(B \to C) \to 0$. We call a sequence exact if it is exact at all interior objects in the sequence, which is equivalent to the exactness of a collection of short exact sequences by the remark above. A functor from one abelian category to another is exact if it transforms exact sequences to exact sequences. Since exactness of a long sequence can be checked by examining shorter sequences involving images and cokernels of morphisms a functor F is exact if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact. Similarly the functor is called left exact if $0 \to F(A) \to F(B) \to F(C)$ is exact, and a similar definition for right exact using only the three arrows on the right of the short exact sequence.

Definition 1. *An R-module B is projective if and only if h_B is an exact functor. It is injective if and only if h'_B is exact.*

Note that h_B is left exact. This is a consequence of the fact that it has a left adjoint so it preserves limits, in particular kernels are preserved. But it is also easy to check directly: if $0 \to L \to M \to N \to 0$ is exact, then applying h_B to the first three terms gives the sequence $0 \to h_B(L) \to h_B(M) \to h_B(N)$. The image of the first homomorphism is 0, the kernel of the second is all elements $\phi \in Hom(B, L)$ for which $\phi(b) \in \ker(L \to M)$, which by exactness of the original sequence means ϕ is the zero homomorphism so the transformed sequence is exact at the first step. At the next step the image is the result of following homomorphisms from B to L by the inclusion homomorphism from L to M. Any homomorphism in the kernel of the map of $h_B(L)$ to $h_B(M)$ must have values in $L = ker(M \to N)$, so exactness holds at this step. Thus h_B is left exact for any module

B. Generally, if $M \to N \to 0$ is an exact sequence of modules, it is not necessarily true that $h_B(M) \to h_B(N) \to 0$ is exact. For example if we consider $\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$ there is a unique $\mathbf{Z}$–module homomorphism from $\mathbf{Z}/2\mathbf{Z}$ to $\mathbf{Z}$ but more than one from $\mathbf{Z}/2\mathbf{Z}$ to itself. So $\mathbf{Z}/2\mathbf{Z}$ is not a projective $\mathbf{Z}$–module.

Similarly, $h'_B(M)$ transforms the exact sequence $L \to M \to N \to 0$ to $0 \to h'_B(N) \to h'_B(M) \to h'_B(L)$ which is exact. So to check that a module M is injective it is sufficient to check that h'_B is left exact. For example the abelian group $\mathbf{Z}$ is not exact since applying $h'_{\mathbf{Z}}$ to the exact sequence $0 \to \mathbf{Z} \to \mathbf{Z}$ given by multiplication by 2 yields $\mathbf{Z} \to \mathbf{Z} \to 0$ with the first map multiplication by 2, which is not exact.

We wish to prove the standard results about projective modules and then dualize to get those about injective modules.

Proposition 1. *Every free module P is projective.*

Proof: Suppose that P is a free R-module with a subset W of P which gives a base. Then $Hom_{\mathcal{R}-\mathcal{MOD}}(P, M)$ is the abelian group which is the product of M with itself $|W|$ times. If $L \to M \to N$ is exact, then so is $L^{|W|} \to M^{|W|} \to N^{|W|}$ by checking componentwise in the products.

It turns out that under some conditions on a ring every projective module is free. This is true if R is a commutative local ring, a principal ideal domain, or $D[x_1, \cdots, x_n]$ for a principal ideal domain D.

We will need to be able to recognize when an exact sequence splits, that is is isomorphic to one of the form $0 \to A \to A \oplus C \to C \to 0$, with the middle maps the inclusion and projection maps for direct sums and products.

Proposition 2. *Let $f : A \to B, g : B \to C$ be morphisms in an abelian category such that the sequence $0 \to A \to B \to C \to 0$ is exact. The sequence is split exact if and only if there exists a morphism $h : C \to B$ such that $gh = \mathrm{id}_C$. This is equivalent to the existence of a morphism $k : B \to A$ such that $fk = \mathrm{id}_A$.*

Proof: If the sequence splits the projection map to the first factor can be take as k and the inclusion of the second factor into the direct product can be taken as h. Conversely, if there exists $h : C \to B$ such that $gh = \mathrm{id}_C$ we can construct an isomorphism $A \oplus C \to B$ by taking $f \oplus h$. The inverse of this isomorphism is the map $B \to \mathrm{image}(f) \oplus \mathrm{image}(h)$ given by $(\mathrm{id} - hg) \oplus hg$ since the first summand is in the kernel of g (which is isomorphic to A) and the second is in the image of C under h which is isomorphic to C. The final statement follows from the fact that the kernel of k is isomorphic to C via the map g, so that we may take the map h to be the inverse of this isomorphism.

It might be worthwhile to give an example of a projective module which is not free. We need some criterion to recognize projective modules.

Proposition 3. *An R-module P is projective if and only for any exact sequence $B \to C \to 0$ any morphism of $P \to C$ comes from a morphism $P \to B$ via composition with $B \to C$. An R-module P is projective if and only if there is an R-module P' such that $P \oplus P'$ is free.*

Proof: The first statement is just a restatement of the condition for h_P to be right exact. If P is projective consider the epimorphism ϕ from a free module $F \to P \to 0$ which is exact.

Hence $Hom(P,F) \to Hom(P,P) \to 0$ is exact and thus there exists a homomorphism $\psi : P \to F$ such that $\phi \circ \psi = 1$. By the remark above the exact sequence splits so that the free module F is isomorphic to the sum of P and the kernel of ϕ. Conversely, if the module P is a direct summand of a free module F, say $P \oplus K = F$ then any homomorphism in $Hom(P,C)$ can be extended to a morphism in $Hom(F,C)$ by mapping K to 0. Since the functor h_F is right exact, it follows that h_P is right exact and hence P is projective.

As a consequence of this proposition we see that direct sums of modules are projective if and only if each summand is projective.

Proposition 4. *A* $\mathbf{Z}$*–module* P *is projective if and only if it is free.*

Proof: Since projective modules are summands of free modules, we need only verify that submodules of free $\mathbf{Z}$–modules are free. For finitely generated abelian groups this follows from the structure theorem. In general, we show that every subgroup P of a free abelian group is free. Let F be a free abelian group, take a basis x_i where i runs over an index set I, which we can well order and and assume that each element of I has an immediate successor which we denote $i+1$ (if $\{j > i\}$ is empty there is no successor, so we extend the index set I by adding in a final element β larger than all elements of I to be able to have successors in J for all elements of I). The finitely generated abelian groups F_j with basis $x_i, i < j$ are free, $P_j = P \cap F_j$ are subgroups of finitely generated free abelian groups, hence free, and $\cup P_j = P, P_{i+1} \cap F_i = P_i$, P_{i+1}/P_i is free since it is a submodule of $F_{i+1}/F_i \simeq \mathbf{Z}$, and the exact sequence $0 \to P_i \to P_{i+1} \to P_{i+1}/P_i - > 0$ splits so that $P_{i+1} \simeq P_i \oplus P_{i+1}/P_i$ is a sum of free modules. hence there are elements $b_i \in P_{i+1}$ such that $P_{i+1} \simeq P_i \oplus \mathbf{Z}b_i$. Then the elements b_i are independent, since in any finite dependency relation let k be the maximal index appearing and consider the relation modulo P_k to derive a contradiction. Further the set b_k with $k < i$ is a basis of P_{i+1}. This shows that the b_k generate all of $P = \cup P_j$, so P is free.

Almost the same argument works over any principal ideal domain to show that any submodule of a free module over a principal ideal domain is free, hence any projective module over a principal ideal domain is free.

Note that over the ring $R = \mathbf{Z}/6\mathbf{Z}$ (which is not a domain and not a local ring, but is a principal ideal ring) the R–modules $P = \mathbf{Z}/3\mathbf{Z}$ and $P' = \mathbf{Z}/2\mathbf{Z}$ are not free R–modules, but $R = P \oplus P'$ as R–modules, so P, P' are projective modules which are not free.

A more interesting example is the domain $D = \mathbf{Z}[\sqrt{-5}]$ which is not a principal ideal domain since it doesn't even have unique factorization as $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$. The ideal generated by $2, 1+\sqrt{-5}$ is a D–module M, which is not free since it is not a principal ideal (both $2, 1+\sqrt{-5}$ are irreducible, and not associate, so there is no common divisor except units). M is the ideal $\{a + b\sqrt{-5} | a, b \in \mathbf{Z}, a+b \text{ even}\}$. The module $M \oplus M$ is isomorphic to $D \oplus D$ via the homomorphism $(r,s) \to (r2 + s(1+\sqrt{-5}), r(1+\sqrt{-5}) + s2)$. To see this observe that since the determinant of $\begin{pmatrix} 2 & 1+\sqrt{-5} \\ 1-\sqrt{-5} & 2 \end{pmatrix}$ is -2 the map is injective, and Cramer's rule shows that it is surjective since given $a + b\sqrt{-5}, c + d\sqrt{-5}$ in M we can compute (r,s) uniquely as solutions of a system of equations. The fact that $M = \{a + b\sqrt{-5} | a, b \in \mathbf{Z}, a+b \text{ even}\}$ guarantees that the (r,s) computed are in $D \times D$.

Isomorphism classes of finitely generated projective modules form a commutative monoid under direct sum, and the Grothendieck group of this monoid is called $K_0(R)$, an interesting group associated to a ring. If all projective modules over R are free, then $K_0(R)$ is isomorphic to $\mathbf{Z}$ via the rank map.

The dual notion to projective modules is that of injective modules. One of the advantages of projective modules over free modules is that the dual notion to free modules does not exist (there is no universal object associated to a set such that giving a map of a module to the set is the same as giving a homomorphsim to the universal object. See Hungerford eercise IV.3.13) . The dual of the first statement of Proposition 3 characterizes injective modules. Every module is a quotient of a projective module, and dually every module is a submodule of an injective module (Hungerford IV.3.12). Direct sums of modules are projective if and only if each is projective, and dually direct products of modules are injective if and only if each is injective. We recall from Definition 1:

Definition. *A module M is injective if and only if the functor h'_M is exact.*

In particular, if M is injective and is a submodule of a module N, the sequence $0 \to M \to N \to M/N \to 0$ is exact, and the exactness of $0 \to h'_M(M/N) \to h'_M(N) \to h'_M(M) \to 0$ implies that there is a homomorphism $k : N \to M$ such that $k \circ i = \mathrm{id}_N$. Then the exact sequence splits as observed above. So an injective module is a direct summand of any module containing it. Conversely, if a module is a direct summand of any module containing it then it is injective (Hungerford Prop. IV.3.13).

All modules over division rings are free, and injective since short exact sequences split. Hence modules over a division ring are both free (hence projective) and injective.

If r is a nonzero divisor the sequence given by multiplication by r is injective: $0 \to R \to R$ is exact , so that $h_J(R) = J \to h_J(R) = J \to 0$ is exact for an injective module J, where the morphism is multiplication by r. Thus an injective module J is divisible by all nonzero divisors r, that is $rJ = J$. We need a criterion to recognize injective modules.

Baer's criterion. *An R-module M is injective if and only if any homomorphism from an ideal J of R to M extends to a homomorphism of R to M.*

Proof: If M is injective consider the exact sequence $0 \to J \to R$ and apply h'_M. Then $h'_M(R) \to h'_M(J) \to 0$ is exact, so every element $\lambda \in \mathrm{Hom}(J, M)$ is the restriction of some element of $\mathrm{Hom}(R, M)$. Conversely, if the criterion is met we must show that whenever K is a submodule of L and $\lambda : K \to M$ is a homomorphism then λ extends to a homomorphism $L \to M$. Consider the set of submodules K' of L together with homomorphisms $\lambda' : K' \to M$. We partially order this set by $(K'/\lambda') \leq (K'', \lambda'')$ if and only if $K' \subset K''$ and λ' is the restriction of λ'' to K'. The set of pairs (K', λ') greater than or equal to (K, λ) is nonempty and has upper bounds to every chain (take the union of domains and the obvious homomorphism to M). By Zorn's lemma we have a maximal element (K', λ') in this set. We show that $K' = L$. If K' is not L, let $l \in L, l \notin K'$. Consider the ideal J of all $r \in R$ such that $rl \in K'$, and homomorphism of J to M given by $\eta(r) = \lambda'(rl)$. By assumption we can extend η to all of R. Define an R-module map on $K' + Rl$ by $\eta'(y + rl) = \lambda'(y) + \eta(r)$, which is well defined since if $y + rl = 0$, then $r \in J$, so that $\eta(r) = \lambda'(rl) = -\lambda'(y)$. Hence $eta'(y + rl) = \eta'(y_1 + r_1 l)$if $y + rl) = y_1 + r_1 l$. Then $(K' + Rl, \eta') > (K', \lambda')$ contradicting the maximality.

This criterion makes it easier to decide when modules are injective.

Proposition 5. *An module M over a principal ideal domain R is injective if and only if for any nonzero $\alpha \in R$, $\alpha M = M$, that is M is divisible.*

Proof: We saw above that injective modules are divisible by non zero divisors, so over a domain R any injective module is divisible by any nonzero elements of R. Conversely, suppose that M is divisible. By Baer's criterion it is enough to show that for any ideal J of R and homomorphism $\lambda : J \to M$ we can find an extension of λ to a homomorphism $\lambda' : R \to M$. If J is zero take $\lambda' = 0$. Otherwise there is a nonzero element $\alpha \in R$ such that $J = R\alpha$. Consider $\lambda(\alpha) \in M$. Since $\alpha M = M$ we have that there exists $m \in M, \alpha m = \lambda(\alpha)$ Define $\lambda'(r) = rm$. This extends λ and hence M is injective.

For example, the fraction field FF of a principal ideal domain R is an injective module, as is the R module FF/R.

Similarly, when R is a nontrivial ring $\mathbf{Z}/N\mathbf{Z}$, the module R is both projective (it is free) and injective (we need to check that for each ideal J of R that any homomorphism of J to R extends to a map of R to R. The nonzero ideals J are principal, generated by integers m dividing N, and any homomorphism takes m to an element of R annihilated by N/m. Such elements are precisely the multiples of m, so we may extend by mapping $1 \in R$ to $\lambda(m)/m$.)