Mathematics 551 **Algebra** **Fall 2006**

Week 7 Factorization and localization
Sections 3.3–3.4

1. Find all ring homomorphisms of the real numbers to itself.
2. Let T be the set of lower triangular 2×2 matrices with integer entries. Determine the ideals of T.
3. Show that if R is a field (a commutative division ring) then a matrix in the ring $M_n(R)$ of $n \times n$ matrices with entries from R is a zero divisor if and only if it is not invertible. Is this true for an arbitrary commutative ring R?
4. Let a, b be elements in a ring R. Show that if $1-ab$ is invertible in R than so is $1-ba$.
5. The Burnside ring of a group G. Let G be a group, and consider the category of finite sets with an action of G (G-sets for short).
   a) Show that if X, Y are finite G-sets than the disjoint union $X+Y$ and the product set $X \times Y$ are also finite G-sets and that $Z \times (X+Y)$ is isomorphic to $Z \times X + Z \times Y$ as G-sets.
   b) Show that the set of isomorphism classes of finite G-sets is a semiring (satisfies all ring axioms except that under $+$ the set is only assumed to be a commutative monoid) under the operations of (a)
   c) The Burnside ring $B(G)$ is by definition the Grothendieck ring of the semiring of finite G-sets. Compute the Burnside ring of the trivial group.
   d) Let H be a subgroup of G. Show that the function defined by $\alpha_U(X) = |X^U|$ which assigns to a G-set X the number of elements of X fixed by U is a ring homomorphism from $B(G)$ to the integers.
   e) Compute the Burnside ring of the group with two elements. Hint: Show that for general G, as an abelian group under $+$ $B(G)$ is a free abelian group on the G-sets G/K as K runs over representatives of subgroups of G up to conjugacy. Study the multiplication law in the special case that G has two elements.
6. Hungerford 3.2.24
7. Hungerford 3.3.4
8. Hungerford 3.4.14