Mathematics 551 **Algebra** **Fall 2006**

Week 4 Structure of finitely generated abelian groups,
Sections 2.1-2.4

1. Let G be the group of invertible 3×3 real matrices M for which left multiplication by M on column 3-vectors permutes the three coordinate lines in R^3. Show that there is a homomorphism from G onto S_3. Use this homomorphism to find normal subgroups $G_1 \subset G_2 \subset G$ of G for which successive quotients are abelian. Identify the groups G, G_1, G_2 in terms of matrices, and explicitly describe the quotients up to isomorphism

2. Let $Quat$ be the subgroup of $GL(2, \mathbf{C})$ generated by $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

   a) Show that $Quat$ has order 8 and every subgroup is normal

   b) Show that $Quat$ is not isomorphic to the dihedral group D_4.

   c) Show that $Quat$ maps homomorphically onto any group of order 8 with precisely 6 elements of order 4.

   d) Show that there are exactly 5 groups of order 8 up to isomorphism

3. Let G be the group generated by x_1, x_2, x_3 subject to relations $x_1x_3 = x_3x_1, x_2x_3 = x_3x_2, x_2x_1 = x_3x_1x_2$.

   a) Show that the set of integer triples (r, s, t) is a group M under the product $(r, s, t)(r', s', t') = (r + r' + st', s + s', t + t')$ and that there is a surjective homomorphism from G onto M.

   b) Show that there is a normal subgroup N of M such that M/N is isomorphic to $\mathbf{Z} \times \mathbf{Z}$.

   c) Show that the groups G and M are isomorphic.

4. Let G be the group generated by a, b with relations $a^2 = 1, b^3 = 1$.

   a) Show that G is infinite and not abelian.

   b) Show that there is a surjective homomorphism of G onto $SL(2, \mathbf{Z})/\{\pm 1\}$

   c) Compute the kernel of your homomorphsim in b).

6. Hungerford II.2.9

7. Hungerford II.4.9

8. Hungerford II.4.15. Show more generally that any nontrivial normal p-group H of a p-group G contains at least p elements of the center of G. Is a normal subgroup H of prime square order in a group of prime power order necessarily in the center?