Mathematics 551 **Algebra** **Fall 2006**

Week 3 Universal objects and free objects
Sections 1.7 – 1.9

1. Let G and H be groups and ϕ_1, ϕ_2 group homomorphisms from G to H. Let $\mathcal{G}$ and $\mathcal{H}$ denote the 1 object categories with morphism sets G and H, and let F_1, F_2 be the functors from $\mathcal{G}$ to $\mathcal{H}$ corresponding to ϕ_1, ϕ_2.
   a) Show that there exists a natural transformation from F_1 to F_2 if and only if there exists an element $y \in H$ such that $\phi_1(g)y = y\phi_2(g)$ for all $g \in G$. Show that any natural transformation of F_1 to F_2 is a natural isomorphism.
   b) Show that when $G = H$ there is a natural transformation from F_1 to the identity functor if and only if ϕ_1 is an inner automorphism of G (that is, conjugation by some element)
   c) Show that there are categories $\mathcal{C}$ and functors F_1, F_2 of the category to itself such that for all objects C there are isomorphisms $F_1(C) \simeq F_2(C)$ but there is no natural transformation of F_1 to F_2.

2. Let $\mathcal{C}, \mathcal{D}$ be categories and let $\mathcal{C}_0, \mathcal{D}_0$ be skeletons of the respective categories. Recall that $\mathcal{C}, \mathcal{D}$ are equivalent categories if and only if they have isomorphic skeletons.

   a) For each object of $\mathcal{C}$ let C_0 be the unique object of $\mathcal{C}_0$ isomorphic to it and let $f_C : C \to C_0$ be an isomorphism. Show that the assignment of an object C to the corresponding C_0 and of a morphism $g : C \to C'$ to the morphism $f_{C'} \circ g \circ f_C^{-1} : C_0 \to C_0'$ is a functor $G_{\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{C}_0$.

   b) Let $F_{\mathcal{C}}$ be the functor considering $\mathcal{C}_0$ as a subcategory of $\mathcal{C}$. Show that the functors $G_{\mathcal{C}} \circ F_{\mathcal{C}}$ and $F_{\mathcal{C}} \circ G_{\mathcal{C}}$ are naturally isomorphic to the identity functors on $\mathcal{C}_0$ and $\mathcal{C}$ respectively. Use this to show that if $\mathcal{C}, \mathcal{D}$ are equivalent categories then there exist functors F from $\mathcal{C}$ to $\mathcal{D}$ and G from $\mathcal{D}$ to $\mathcal{C}$ such that the functors $G \circ F$ and $F \circ G$ are naturally isomorphic to the identity functors on $\mathcal{C}$ and $\mathcal{D}$ respectively. Sometimes this property is taken as the definition of equivalent categories, which has the advantage of giving the functor establishing the equivalence.

   c) Conversely, show that if there exist functors F from $\mathcal{C}$ to $\mathcal{D}$ and G from $\mathcal{D}$ to $\mathcal{C}$ such that the functors $G \circ F$ and $F \circ G$ are naturally isomorphic to the identity functors on $\mathcal{C}$ and $\mathcal{D}$ respectively then $\mathcal{C}$ and $\mathcal{D}$ have isomorphic skeletons.

3. Let **MAT** be the category which has as object set the set of nonnegative integers and the set of morphisms from n to m is the set of $m \times n$ real matrices, with composition the usual matrix product (in case n or m is 0, we agree that there is a single $m \times n$ matrix which we denote $0_{m,n}$ and its product with any n by p matrix is $0_{m,p}$) . Show that the category **MAT** is equivalent to the category of finite dimensional real vector spaces.

4. Let D_4 be the group of rotations and reflections mapping the square with vertices the 4th roots of unity to itself. Let $G = Aut(D_4)$ be the group of automorphisms of D_4.

    a) Show that D_4 contains exactly two elements of order 4.

    b) Show that there is an inner automorphism of D_4 (ie, conjugation by some element z in D_4) which interchanges the order 4 elements.

    c) Compute the order of G by studying the orbit and stabilizer of an order 4 element of D_4 under the action of G.

    d) Are the groups D_4 and G isomorphic?

5. Let G_1, G_2 be simple groups. Show that every normal subgroup of $G_1 \times G_2$ which is not trivial or all of $G_1 \times G_2$ is isomorphic to G_1 or G_2.

6. Hungerford I.7.6

7. Hungerford I.8.2

8. Hungerford I.8.7