Mathematics 551 **Algebra** **Fall 2006**

Week 10 Modules over principal ideal Domains, Adjoint Functors
Sections 4.4-4.7, 10.2

1. Let R be a ring. Consider the forgetful functor from the category of (left) R-modules to abelian groups. Determine the right adjoint to this functor (Hint: think about what the morphisms of the abelian group of the R-module R to an abelian group A must be if a right adjoint exists).

2. We say that a category **C** is additive if
   1. The set of morphisms between two objects is an abelian group, and the maps obtained by composition are abelian group homomorphisms.
   2. There is a zero object 0 such that the morphisms of 0 to itself form the trivial group.
   3. For any pair of objects X_1, X_2 there is an object Y and morphisms $p_i : Y \to X_i$ and $j_i : X_i \to Y$ such that $p_i j_i = 1_{X_i}, p_i j_k = 0$ if $i \neq k, j_1 p_1 + j_2 p_2 = 1_Y$.

   Show that the category of modules over a ring is an additive category. Is the category of all commutative rings an additive category?

3. A functor F from an additive category **C** to an additive category **D** is said to be additive if for all objects X, Y of **C** the map that the functor induces from $Mor(X, Y)$ to $Mor(FX, FY)$ is a homomorphism of abelian groups. Show that the forgetful functor from modules to abelian groups is additive. Show that if F is additive, then any left or right adjoint of F is also additive.

4. Let $D = \mathbf{Z}[i]$ be the ring of Gaussian integers. Let K be the submodule of D^3 generated by $(1, 3, 6), (2 + 3i, -3i, 12 - 18i), (2 - 3i, 6 + 9i, -18i)$. Determine the quotient D–module D^3/K as a product of cyclic D–modules. Is it a finite set?

5. Hungerford 4.4.11

6. Hungerford 4.5.2

7. Hungerford 4.6.6

8. Hungerford 10.2.3

9. Hungerford 10.2.4