Mathematics 551 **Algebra** **Fall 2006**

Text Hungerford, *Algebra* , GTM 73

References Lang, *Algebra*
Jacobson, *Basic Algebra I*
MacLane, *Categories for the Working Mathematician*
van der Waerden, *Modern Algebra*
Kostrikin and Shafarevitch, *Algebra I*

Week 1 Basics of categories, functors and groups
Sections 1.7, 1.1 – 1.3

1. Let S be a set and define a product on S by $ab = b$. Show that S is a semigroup. Under what conditions does S contain an identity element?
2. Show that any finite group of even order contains at least two elements solving $x^2 = 1$
3. Prove or disprove: The additive group $\mathbf{Q}$ of all rational numbers is isomorphic to the mutliplicative group $\mathbf{Q}^*$ of nonzero rationals.
4. Determine the center of the symmetric group S_n on n letters for $n = 0, 1, 2, \ldots$
5. Which of the following subsets of $G = GL(n, \mathbf{R})$ are subgroups? Explain.
   a) $SL(n, \mathbf{R})$, the n by n matrices of determinant 1
   b) the upper triangular matrices in G
   c) the matrices in G with sum of diagonal entries 0
   d) the orthogonal matrices in G
   e) the symmetric matrices in G
   f) the matrices in G with last row $(0, 0, \ldots, 1)$
   g) the matrices in G with positive eigenvalues
6. Find the centers of $GL(n, \mathbf{R}), SL(n, \mathbf{R})$.
7. Show that assigning a group to its center is not a functor from the category of groups to the category of abelian groups.
8. Let G be a group and let $Z(G)$ denote the center of G and $Aut(G)$ the group of automorphisms of G. Show that if $\sigma \in Z(Aut(G))$ is a central automorphism then $\sigma(g)$ and g determine the same coset modulo $Z(G)$. In particular, determine $Z(Aut(G))$ when $Z(G)$ is trivial.