Mathematics 551 **Algebra** **Fall 2006**

Colimits and limits of Functors

We considered a general construction in class that began with a category $\mathcal{C}$ (which will have objects we are interested in) and functor F from another category $\mathcal{J}$ to $\mathcal{C}$ (which will often be thought of as indexing the objects $F(j)$ by objects in J).

Construct a new category $\mathcal{D}$ in which the objects consist of an object C of $\mathcal{C}$ together with morphisms in $\mathcal{C}$, say $\phi_j : F(j) \to C$ for each object j in $\mathcal{J}$. We further require that if $\psi : j \to k$ is a morphism in J then $\phi_j = \phi_k \circ \psi$. We think of an object of $\mathcal{D}$ as an object C of $\mathcal{C}$ together with a compatible family of morphisms from $F(j)$ to C. The reader may prefer to picture such objects via diagrams, but in these notes we will not draw a single commutative diagram, choosing instead to name the morphisms and state explicitly which compositions equal which other morphisms.

The morphisms of such an object D to another object D' specified by an object C' of $\mathcal{C}$ together with morphisms $\phi'_j : F(j) \to C'$ is a morphism $\eta : C \to C'$ such that $\phi'_j = \phi_j \circ \eta$. Two such morphisms $D \to D', D' \to D''$ clearly compose to give a morphism, and $\mathcal{D}$ is a category.

We call an initial object of $\mathcal{D}$ a *colimit of the functor* F. Colimits occur so often that many have special names, which will be described in the remarks below. A colimit consists of an object $colim(F)$ in $\mathcal{C}$ together with morphisms $\phi_j : F(j) \to colim(F)$ for each object in J such that whenever $\psi : j \to k$ is a morphism in J then $\phi_j = \phi_k \circ \psi$. In brief, to give a compatible family of morphisms to any object is to give a morphism of the colimit object to the object.

We often abuse terminology and also refer to the object C in $\mathcal{C}$ underlying the initial object as the colimit of F, but of course we need to remember that it comes with the morphisms $\phi_j : F(j) \to C$ which define the object in $\mathcal{D}$.

Examples of colimits:

1. Suppose that $\mathcal{J}$ is the empty category with no objects and no morphisms. Then a functor from $\mathcal{J}$ to $\mathcal{C}$ contains no information. The category $\mathcal{D}$ above is isomorphic to the original category $\mathcal{C}$, so the limit of such a functor is an initial object of $\mathcal{C}$.

2. Suppose that $\mathcal{J}$ is a category with a single object and a single morphism. Then a functor F from $\mathcal{J}$ to $\mathcal{C}$ is a choice of an object C_0 in $\mathcal{C}$. An object in $\mathcal{D}$ is an object C of $\mathcal{C}$ together with a morphism $\phi_0 : C_0 \to C$. A morphism in $\mathcal{D}$ between objects $\phi_0 : C_0 \to C, \phi'_0 : C_0 \to C'$ is a morphism $\psi : C \to C'$ such that $\phi'_0 = \psi \circ \phi_0$. Hence the colimit of this functor F is given by an object $colim(F)$ which is an object of $\mathcal{C}$ furnished with a morphism $\phi : C_0 \to colim(F)$ such that any morphism of C_0 to an object of $\mathcal{C}$ factors uniquely through ϕ. The colimit in this case is given by the morphism $1_{C_0} : C_0 \to C_0$, so it always exists, even if it is not very interesting.

3. Suppose that $\mathcal{J}$ is the discrete category of a set of objects where the only morphisms are the identity morphism of each object to itself. A functor F from this $\mathcal{J}$ to $\mathcal{C}$ is a choice

of objects C_j of $\mathcal{C}$, indexed by the objects of $\mathcal{J}$. The category $\mathcal{D}$ then consists of objects C of $\mathcal{C}$ together with morphisms $\phi_j : C_j \to C$. A colimit for this functor is called the *coproduct* of the objects C_j. If it exists it is an object $coprod(C_j)$ together with morphisms $\phi_j : C_j \to C$ such that to give morphisms from C_j to an object C' is to give a morphism from $coprod(C_j)$ to C'. The notation $\coprod_{j\in\mathcal{J}} C_j$ is often used for the coproduct in a category.

The categories $\mathcal{J}$ in 1., 2. are discrete, so are special cases of coproducts. In particular, if the empty coproduct exists it is an initial object in $\mathcal{C}$. For more complicated $\mathcal{J}$ the coproduct may not exist. We showed in class that the disjoint union of sets together with the inclusions of each set into the disjoint union is the coproduct in the category of sets.

4. The coproduct of an arbitrary set of abelian groups exists in the category of abelian groups. We write abelian group laws as $+$, abelian group identities as 0. Given abelian groups A_j let $\oplus_j A_j$ denote the abelian group of functions on the index set of objects in $\mathcal{J}$ which have $f(j) \in A_j$ and for all but finitely many values of j the value $f(j)$ is the group identity 0 of A_j, with the abelian group law given by addition of function values. The morphisms $\phi_j : C_j \to \oplus_j C_j$ are given by sending an element c of C_j to the function which takes value c a j and 0 elsewhere. This satisfies the universal property for a coproduct, since given any set of morphisms $\phi'_j : A_j \to A'$ of abelian groups we can define a morphism $\eta : \oplus A_j \to A'$ by sending a function $f(j)$ on the index set of objects to the sum of $\phi'_j(f(j))$ over the finitely many j for which $f(j)$ is not the identity element $0 \in A_j$. This finite sum of elements in an abelian group is defined and independent of the order in which the sum is made. We have $\eta(f_1+f_2) = \sum_j \phi'_j(f_1(j)) + \phi'_j(f_2(j)) = \eta(f_1)+\eta(f_2)$ since ϕ'_j are homomorphisms with values in the abelian group A'. Further, η is the unique such homomorphism from $\oplus A_j \to A'$ satisfying $\eta \circ \phi_j = \phi'_j$.

5. Let $\mathcal{J}$ be the category formed by a partially ordered set J of objects with $Mor(j,k)$ containing a single morphism if $j \le k$ and empty otherwise, with the obvious composition law. A functor F from this category to a category $\mathcal{C}$ is given by objects C_j indexed by the partially ordered set, together with morphisms $\phi_{jk} : C_j \to C_k$ for $j \le k$ such that ϕ_{jj} is the identity morphism and $\phi_{kl} \circ \phi_{jk} = \phi_{jl}$ when $j \le k \le l$. Such objects are often called a *direct system* of objects and morphisms. The colimit of this functor is called the direct limit of the system C_j, usually denoted by $\varinjlim C_j$. To give morphisms from the objects C_j to an object C' which are compatible with the ordering is equivalent to giving a morphism of $\varinjlim C_j$ to C'.

For example, in the category of sets suppose that we have a directed system ordered by inclusion. The direct limit is just the union of the sets. In the category of groups the direct limit of a direct system G_j of groups can be constructed as the set G of functions f on the partially ordered set J such that $f(j)$ is in G_j, and for $j \le k$, $f(k) = \phi_{jk}(f(j))$. The group structure is inherited from the fact that the values of such functions at a point are in a group. There are homomorphisms $\psi_j : G_j \to G$ given by assigning an element a of G_j to the function which has value at k equal to $\phi_{jk}(a)$ at $j \le k$ and the identity element of G_k otherwise. The universal property to be checked is that if $\alpha_j : G_j \to H$ is a compatible family of group homomorphisms, then there is a unique homomorphism $\eta : G \to H$ such that $\alpha_j = \eta \circ \psi_j$. The homomorphism is given by $\eta(f) = \alpha_j(f(j))$, which is well defined since $\alpha_k \circ \phi_{jk} = \alpha_j$ when $j \le k$ by the compatibility of the direct system homomorphisms. For example the direct limit of the groups of p^n roots of unity ordered by inclusion is the group of all p-power roots of 1. The direct limit of the system $\mathbf{Z}/p^j\mathbf{Z}$ with homomorphisms ϕ_{jk} given by sending 1 to p^{k-j} is the group $Z(p^\infty)$ of the homework problem I.4.7 in Hungerford (and isomorphic to the group in the previous sentence).

6. Suppose that $\mathcal{J}$ has two objects and two morphisms from the first object to the other. To give a functor F from $\mathcal{J}$ to $\mathcal{C}$ is to give objects C_1, C_2 of $\mathcal{C}$ and two morphisms $\alpha, \beta : C_1 \to C_2$. The colimit of this functor is an object C furnished with morphisms $\phi_1 : C_1 \to C, \phi_2 : C_2 \to C$ such that $\phi_2 \circ \alpha = \phi_2 \circ \beta = \phi_1$ (hence ϕ_1 is determined by ϕ_2) such that for any $\phi'_1 : C_1 \to C', \phi'_2 : C_2 \to C'$ such that $\phi'_2 \circ \alpha = \phi'_2 \circ \beta = \phi'_1$ there is a unique homomorphism $\gamma : C \to C'$ with $\gamma \circ \phi_i = \phi'_i$ for $i = 1, 2$. In other words, to give a morphisms of C_2 to an object C' such that compositions with α and β lead to the same result is to give a homomorphism of the colimit to C'. This colimit is called the *coequalizer* of the two morphisms.

In the case that C_1, C_2 are groups and α, β group homomorphisms the coequalizer of α, β would be a group G together with homomorphisms $\phi_2 : C_2 \to G$ such that $\phi_2 \circ \alpha = \phi_2 \circ \beta$ such that G has a morphism to any other group that the C_2 map to in such a way that results of composing with α, β are the same. For example, to construct the coequalizer when β is trivial is to give a group G, and a homomorphism of C_2 to G which is trivial on the image of α, and universal in the sense that any other homomorphism of C_2 to a group which is trivial on the image of α factors through G. If N is the normal subgroup generated by the subgroup $\alpha(C_1)$ in C_2 then the quotient group C_2/N together with the quotient map $\pi : C_2 \to C_2/N$ satisfies the univeral property. This group is usually called the cokernel of the homomorphism α.

Limits of Functors

If we start with a functor F from $\mathcal{J}$ to $\mathcal{C}$ there is a functor F^{op} from $\mathcal{J}^{op}$ to $\mathcal{C}^{op}$ obtained by reversing the arrows in diagrams describing the functorial nature of F. The colimit of F^{op} is usually called the limit of the functor F. This is equivalent to defining the limit of F to be a terminal object of the category with objects consisting of objects C of $\mathcal{C}$ together with morphisms $\phi_j : C \to F(j)$ such that when $\psi : j \to k$ then $\psi \circ \phi_j = \phi_k$. As before the morphisms of such objects are morphisms of C to C' compatible with the ϕ_j, ϕ'_j. Hence a limit of the functor F is given by an object $lim(F)$ of $\mathcal{C}$, together with morphisms $\pi_j : lim(C) \to F(j)$ such that any compatible morphisms $C' \to F(j)$ come by composing a morphism to $\lim(F)$ with the π_j. In brief, to give a compatible family of morphisms from an object is to give a morphism to the limit object. The notion is dual to that of the earlier sections, and all examples there can be dualized.

7. The standard example is to take a discrete category $\mathcal{J}$, that is an indexed set of objects C_j. If the limit exists it is called the product of the objects. The universal property of the product is that given morphisms of an object C' to all C_j , there is a unique morphism from C' to the product $\prod C_j$ such that the morphisms are the composition of this morphisms with the projections.

As expected, the usual product of sets is the categorical product in the category Set. For categories in which the objects are sets with extra structure, it is often the case that the product is the product of the sets endowed with extra structure. The category of cyclic groups does not have products in general.

8. The dual of the notion in section 5. is the *inverselimit* of an inverse system (sometimes called the projective limit). It is traditional to take the opposite of the category $\mathcal{J}$ given in that section, that is an inverse system is a family of objects C_j together with morphsims $\phi_{ij} : C_j \to C_i$ for $i \leq j$ such that ϕ_{jj} is the identity and $\phi_{ik} = \phi_{ij} \circ \phi_{jk}$. The inverse limit $\lim\limits_{\leftarrow} C_j$ is an object together with morphisms $\psi_k : \lim\limits_{\leftarrow} C_j \to C_k$ such that $\phi)ij \circ \psi_j = \psi_i$ when $i \leq j$. Any other such compatible family of morphsims to the inverse system is obtained via a homomorphism to the inverse limit followed by the ψ_k.

In the category of groups the inverse limit can be constructed by taking the subgroup of the product consisting of tuples $(\ldots, c_j, \ldots)$ where the obivous relation is required between c_j, c_k when $j \leq k$. The morphisms to the groups in the inverse system are the projections.

For example, the case of the directed system of groups $\mathbf{Z}/p^j\mathbf{Z}$ with homomorphisms ϕ_{ij} given by reduction modulo p^i gives a limit denoted by $\mathbf{Z}_p$ called the p-adic integers. Note that contrary to the example in section 5 this group has no elements of finite order.

9. The dual of the objects in section 6. are called the equalizer of the morphisms α, β, which is an object C together with a morphism $\phi : C \to C_1$ such that $\alpha \circ \phi = \beta \circ \phi$ which satisfies a univeral property among such objects. To give a morphism to C_1 for which compositions with α, β are equal is to give a morphism to the equalizer. Foe example, in the category of groups the equalizer of a homomorphism $\alpha : G_1 \to G_2$ and the trivial homomorphism is the kernel of α, furnished with the inclusion in G_1. Any homomorphism to G_1 which composed with α is trivial is in the kernel of α.