Mathematics 551 **Algebra** **Fall 2006**

Dimension of spaces of intertwining operators

Let F be a field and let V, W be finite dimensional vector spaces over F. Suppose that $L \in \mathrm{End}_F(V), L' \in \mathrm{End}_F(W)$. A vector space map $\phi : V \to W$ intertwines L and L' if and only if $\phi L = L'\phi$. The set of intertwining maps is a vector space $I_{L,L'}$ over F. For example if $V = W, L = L'$ then $I_{L,L'}$ is the vector space of endomorphisms of V commuting with L.

We can recast the definition of $I_{L,L'}$ in terms of modules over a PID. Let $R = F[x]$ be the PID of polynomials in x with coefficients in F. We can consider V as an R–module by defining $xv = Lv$. Similarly W is an R–module by defining $xv = L'v$. Then the vector space $I_{L,L'}$ equals $\mathrm{Hom}_R(V, W)$, so it may be analyzed by module theoretic techniques.

Since V, W are torsion modules over R there exist polynomials $g_i, h_j \in R$ such that $V \simeq \oplus R/(g_i), W \simeq \oplus R/(h_j)$. By the fact that finite direct sums of modules are the same as finite products of that module and the defining properties of direct sums and products we have $\mathrm{Hom}_R(V, W) \simeq \oplus_{i,j}\mathrm{Hom}_R(R/(g_i), R/h_j)$ so the problem is reduced to computing the vector space $\mathrm{Hom}_R(R/(g_i), R/(h_j))$.

The homomorphisms of a cyclic module R/g to any module N are determined by the image of a generator, which can be taken to be any element of $n \in N$ such that $gn = 0$.

Lemma. *Let d be the greatest common divisor of g_i, h_j so that $(g_i, h_j) = (d)$. Then* $\mathrm{Hom}_R(R/(g_i), R/(h_j)) \simeq (h_j/d)R/(h_j)$.

Proof: The elements of $n \in R/(h_j)$ such that $g_i n = 0$ are represented by polynomials $p(x)$ modulo h_j which satisfy that $p(x)g_i$ is a multiple of h_j. Those $p(x)$ are those which are a multiple of h_j/d.

Since $(h_j/d)R/(h_j)$ is the kernel of the quotient map from $R/(h_j)$ to $R/(h_j/d)$ it has dimension as a vector space over F which equals $\deg(h_j) - \deg(h_j/d) = \deg(d)$. This shows that

$$\dim_F I_{L,L'} = \sum_{i,j} \deg\ \gcd(g_i, h_j)$$

In the special case that $V = W, L = L'$ we obtain a formula for the vector space of all linear maps commuting with L.

Theorem. *Let V be an n-dimensional F-vector space and let L be an F-linear map of V to itself. Then* $\dim_F I_{L,L'} = \sum_1^n (2n - 2i + 1)\deg\ g_i$

Proof: The previous result gives that the dimension is $\sum_{i,j} \deg\ \gcd(g_i, g_j)$. Since g_i divides g_{i+1} this greatest common divisor is $g_{\min i,j}$. The sum $\sum_{i,j} \deg\ g_{\min i,j}$ equals $\sum_j (\sum_{i=1}^j \deg\ g_i + \sum_{i=j+1}^n \deg\ g_j)$ in which the term $\deg\ g_k$ appears $(n-k+1)+(n-k)$ times.

For example, all polynomials in L commute with L, forming a subspace of dimension equal to the degreed of the minimal polynomial of L. The full space of commuting transformations exceeds this dimension by a positive amount if and only if some invariant factor g_j for $j < n$ is not constant, that is if and only if the characteristic polynomial has degree greater than the minimal polynomial. Hence the characteristic polynomial of L equals the minimal polynomial if and only if any transformation commuting with L is a polynomial in L.

For a second example we determine the matrices commuting with the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Since this matrix is in rational canonical form the invariant polynomials are $g_1 = x - 1, g_2 = (x-1)^2$ so that the dimension of the space of matrices commuting with this one is (6-4+1)+(6-6+1)2=5. The proof of the theorem above shows that the linear transformations commuting with the transformation defined by the matrix above are $F[x]$-module endomorphisms of $F[x]/(x-1) \oplus F[x]/(x-1)^2$. Using the basis $1; 1, x$ we see that a module map must map $1 \in F[x]/(x-1)$ to an element of form $(a, b(x-1))$, and $1 \in F[x]/(x-1)^2$ to an element of the form $(c, d+ex)$ and hence x to $(c, -e+(d+2e)x)$. Thus the matrices commuting with the matrix above are those of the form

$$\begin{pmatrix} a & -b & b \\ c & d & e \\ c & -e & d+2e \end{pmatrix}.$$