Mathematics 551 Algebra Fall 2006

Final Problem Set in Lieu of Final Exam

Due 11:59 AM, Dec 21, 2006

NOTE: Check back at this page to see if any corrections have been made. Any changes will be noted below.

VERSION: 11:59 AM Dec. 11

CHANGES:

Added $n > 0$ in problems 8 (12:04 AM 12/14/06)

Changed b) to a) in last part of problem 2 (11:31 PM 12/14/06)

Added "with entries in a field" to problem 7 (11:41 PM 12/14/06)

Removed "Let $F \subset E$ be fields" from problem 2 (9:45 PM 12/19/06)

Added clarification to Problem 10d: ($R[x]$ denotes the ring generated by R and x in B) (10:10 PM 12/19/06)

Don't discuss the problems with others besides me. You may use any books you wish. Prove everything starting from theorems proved in the text or in class. Explain your work,

1. Let N be the integer equal to your student id number. How many isomorphism classes of abelian groups of order N are there? How many isomorphism classes of abelian groups of order N^2 are there? Are there any nonabelian groups of order N? Is there a simple group of order N? (Try to do these last two parts with the Sylow theorems and methods developed in class. If necessary, one can refer to the classification of simple groups in the literature.

2.
   a) Find all possible Jordan forms for complex matrices having characteristic polynomial $(\lambda^2-2\lambda+4)^2$.(List nonsimilar Jordan forms such that every complex matrix with this characteristic polynomial is similar to a matrix in your list.
   b) Of the Jordan forms in a), which are possible Jordan forms of a real matrix with characteristic polynomial $(\lambda^2 - 2\lambda + 4)^2$?

3. Let A be the set of 3–tuples of integers (a,b,c) such that a+b-c =0. Let B be the subgroup of A consisting of tuples in A with a-b+3c divisible by 4. Find the index of B in A.

4. Let G be generated by x,y with relations $x^4 = 1, y^3 = 1, yx = xy^{-1}$. Describe the conjugacy classes of G.

5. Let F be a finite field with an odd number q elements. Let G be the group of two by two matrices with entries in F which have determinant 1. Let $f(\lambda) = \lambda^2 - u\lambda + 1 \in F[\lambda]$.
   a) Prove that if $f(\lambda)$ has two distinct roots in F then the set of all matrices in G with characteristic polynomial $f(\lambda)$ forms a conjugacy class in G, and find the number of elements in this class.
   b) Prove that if $f(\lambda)$ has two equal roots in F then the set of all matrices in G with characteristic polynomial $f(\lambda)$ is the union of three conjugacy classes in G, and find the number of elements in each class.

c) Suppose that $f(\lambda)$ has no roots in F. Find the centralizer of the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & u \end{pmatrix}$$

in G and compute the size of the conjugacy class of this element in G.

d) When F has 7 elements, write the class equation for the conjugation action of G on itself, Do the same for the group G/Z, where Z is the subgroup of scalar matrices in G. Show that G/Z is simple by examining the class equation.

6. Classify (up to isomorphism) all commutative rings R which contain the complex field as a subring and form a dimension 2 vector space over the complex numbers. Extra credit: Same problem with 2 replaced by 3.

7. Show that two diagonalizable matrices A, B with entries in a field can be simultaneously diagonalized (that is both PAP^{-1}, PBP^{-1} are diagonal) if and only if $AB = BA$.

8. Suppose that A is a commutative ring with a finite number $n > 0$ of zero divisors. Show that A has at most $(n+1)^2$ elements.

9. Let H be a subgroup of finite index in a group G. Suppose that G is the union of the conjugates of H. Prove that H=G.

10. Let F be a field of characteristic p. Let B be the ring of polynomials with coefficients from F in noncommuting variables x, y such that $xy - yx = 1$.
    a) Use the commutation relation successively to express commutators of powers of x times y in terms of x (and the analogous statement with x, y interchanged). Show that the center R of the ring B is the ring of polynomials in x^p, y^p with coefficients in F and that B is finitely generated as an R module.
    b) Show that the ring B has no zero divisors.
    c) Let the fraction field of R be K. Show that $B \otimes_R K$ has center K and no zero divisors.
    d) Suppose that p=2. Let L be the fraction field of the commutative subring $R[x]$ of B ($R[x]$ denotes the ring generated by R and x in B). Show that $B \otimes_R K$ is a two dimensional vector space over L (acting on B on the right) and by considering the matrix of left multiplication by $b \in B$ find a homomorphism of B to the ring of 2×2 matrices over L. Use this to show that $B \otimes_R K$ is a division algebra by reducing to computing the inverse of a 2×2 matrix.