Mathematics 551 **Algebra** **Fall 2006**

Counting the number of orbits in a group action

The lemma of Cauchy, Frobenius and Burnside

Let a finite group G act on a finite set X and let X/G be the set of orbits. The formula to be proved is that the number of orbits is the average number of fixed points taken over the elements of the groups:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

where $X^g = \{x \in X | gx = x\}$ is the fixed point set of g.

This formula is often called Burnside's lemma (1900), even though it was known to Cauchy (1845) and Frobenius (1887). Consequently, it is sometimes referred to as the Cauchy-Frobenius lemma or"the lemma which is not Burnside's".

The proof follows from the fact that the orbit of $x \in X$ is the coset space of G_x, stabilizer subgroup of x. Consider the set $\{(x, g) \in X \times G | gx = x\}$. The cardinality of this set can be computed by summing the number of elements with second coordinate g, that is $\sum_{g \in G} |X^g|$. On the other hand, summing over elements with first coordinate x gives $\sum_x |G_x|$. Since the order of the orbit Gx of x under G is $|G|/|G_x|$ we have

$$\sum_{g \in G} |X^g| = \sum_{x \in X} |G|/|Gx|.$$

The right hand summands $|G|/|Gx|$ are constant on orbits and sum to $|G|$ on each orbit. This establishes the formula.

Note that conjugation of the fixed point set X^g by h equals the fixed point set X^{hg}. In applications of the formula, the summands over G are constant on conjugacy classes.

Traditional examples are to coloring problems, that is the set X is the set of functions f from a G-set Y to a set C (such is function is thought of as coloring y by $f(y)$). Note that X^g in this example depends only on the number of cycles of g considered as a permutation of the set Y : the cycle orbits all have the same color, and can be colored independently. Hence if g is a product of k disjoint cycles when acting on Y, $X^g = |Y|^k$.

Example: Suppose that beads of n colors are available to make a necklace of 7 beads by equally spacing beads on a circle. How many truly distinct such necklaces can be made if we regard necklaces to be colored the same if a rigid motion in 3-space takes one to the other.

We may consider this problem as counting the number of ways to color the vertices of a regular 7-gon centered at the origin when colorings in the same orbit under the dihedral group are considered the same. Apply the lemma above to the set X of colorings of the regular 7-gon by n colors, a set with n^7 elements. The number of orbits is the average number of fixed points. The identity element of D_7 fixes n^7 elements (it is a product of 7 1-cycles). The 7 reflections in D_7 are all conjugate and reflection in the x-axis fixes the n^4 colorings where the 4 vertices in the upper half plane are colored arbitrarily and mirrored to color the vertices below (a reflection is a product of 3 2-cycles and 1 1-cycle). The 6 elements of D_7 of order 7 (the nontrivial rotations) fix only the n colorings with all colors the same (such a rotation permutes all vertices cyclically). So the average number of fixed points is $(n^7 + 7n^4 + 6n)/14$, giving that there are $(n^7 + 7n^4 + 6n)/14$ such necklaces.

Similarly, to count colorings of the regular tetrahedron up to rotations we use that the set of rotations of the regular tetrahedron centered at the origin is a group isomorphic to the alternating group A_4 (the 4 lines joining a vertex to the midpoint of an opposite edge are permuted) of order 12. So the orbits in the set of n^4 possible face colorings are the average number of fixed points. The rotations of the tetrahedron consist of 1 identity fixing all n^4colorings, 8 order 3 rotations by $\pm 2pi/3$ around an axis joining a vertex to the central point of opposite faces, fixing n^2 colorings, 3 order 2 rotations about axes joining midpoints of opposite edges, fixing n^2 colorings. So the average is $(n^4 + 8n^2 + 3n^2)/12 = (n^4 + 11n^2)/12$.