Mathematics 551 **Algebra** **Fall 2006**

Brief notes on adjoint functors

"Perhaps the most successful concept of category theory is that of adjoint functor."
Joy of Cats by Adamek, Herrlich and Strecker, page 394

There are several different ways to define adjoint functors. The purpose of these notes is to give a light introduction to the concept. For examples of how heavy the going can become in this subject refer to Chapter V of the tome quoted from above.

We begin with one definition and then develop alternate characterizations.

Let F be a functor from category $\mathcal{C}$ to $\mathcal{D}$.

Definition. *The functor F has a left adjoint if for each object D of $\mathcal{D}$ the functor assigning to an object C of $\mathcal{C}$ the set $\mathrm{Mor}_{\mathcal{D}}(D, F(C))$ is a representable functor from $\mathcal{C}$ to the category of sets.*

Observe that for fixed object D of $\mathcal{D}$ the assignment of an object C of $\mathcal{C}$ to the set the set $\mathrm{Mor}_{\mathcal{D}}(D, F(C))$ is functorial in C : if $\phi : C \to C'$ is a morphism in $\mathcal{C}$ then composition with the morphism $F(\phi) : F(C) \to F(D)$ maps the set $\mathrm{Mor}_{\mathcal{D}}(D, F(C))$ to $\mathrm{Mor}_{\mathcal{D}}(D, F(C'))$.

1. The forgetful functor from the category of groups to the category of sets has a left adjoint, since for each set X the functor sending a group G to $\mathrm{Mor}_{\mathrm{Set}}(X, G)$ is represented by the free group on the set X. More generally, a functor from a category to the category of sets has a left adjoint if and only if there exists a free object in the category attached to each set in the sense of Hungerford Definition I.7.7.

2. Let's determine when the constant functor from a category $\mathcal{C}$ to a category $\mathcal{D}$ with one object D and one morphism has a left adjoint. This is true if and only if the functor assigning an object of C to the one element set $\mathrm{Mor}_{\mathcal{D}}(D, F(C))$ is representable. This holds if and only if there is an object C_0 in $\mathcal{C}$ such that $\mathrm{Mor}_{\mathcal{C}}(C_0, C)$ is a one element set, that is there is a unique morphism from C_0 to any other object in $\mathcal{C}$. Thus the constant functor to a 1 object, 1 arrow category has a left adjoint if and only if there is an initial object in the category.

3. Any functor F from $\mathcal{C}$ to $\mathcal{D}$ which gives an equivalence of categories has a left adjoint, since if G is the functor from $\mathcal{D}$ to $\mathcal{C}$ such that $G \circ F$ and $F \circ G$ are naturally isomorphic to the relevant identity functors we can represent the functor assigning to an object C of $\mathcal{C}$ the set $\mathrm{Mor}_{\mathcal{D}}(D, F(C))$ by the object $G(D)$ (apply the functor F to morphisms in $\mathrm{Mor}(G(D), C)$ to obtain a morphism in $\mathrm{Mor}(FG(D), F(C))$ and use that FG is naturally isomorphic to the identity functor). The same logic show that in this situation the functor G has a left adjoint.

4. Let $\mathcal{C}$ be the category formed from a partially ordered set S and such that $\mathrm{Mor}_{\mathcal{C}}(a,b)$ is a one element set if $a \le b$ and empty otherwise. Let $\mathcal{D}$ be the category formed from another partially ordered set T. A functor F from $\mathcal{C}$ to $\mathcal{D}$ is just an order preserving map which will also be noted by F from the set S to T. Such an F has a left adjoint if and only if for each element $d \in T$ the functor assigning c to $\mathrm{Mor}(d, F(c))$ is representable, that is there is an element $G(d)$ in S such that $\mathrm{Mor}_{\mathcal{D}}(d, F(c))$ is bijective with $\mathrm{Mor}_{\mathcal{C}}(G(d), c)$. This translates to: for each $d \in T$ there is an object $G(d) \in S$ such that for all $c \in S$ we have $d \le F(c)$ if and only if $G(d) \le c$. If so, the set of all $c \in S$ such that $d \le F(c)$ clearly has smallest element $G(d)$, and conversely if this set has a smallest element F has a left adjoint. In this example, the functors having left adjoints are just those for which there is a smallest element among the set of c with $F(c) \ge d$. This example overlaps with a previous one when T is a 1 element partially ordered set, for then F has a left adjoint if and only if S has a minimal element.

Notice that if F has a left adjoint, then we obtain for each object D of $\mathcal{D}$ an object $G(D)$ of $\mathcal{C}$ which represents the functor in the definition, so in particular $\mathrm{Mor}_{\mathcal{C}}(G(D), C)$ is bijective with $\mathrm{Mor}_{\mathcal{D}}(D, F(C))$. This assignment G is a functor from $\mathcal{D}$ to $\mathcal{C}$ since if $g : D \to D'$ is a morphism of $\mathcal{D}$ we obtain a map of sets $\mathrm{Mor}(D', F(C)) \to \mathrm{Mor}(D, F(C))$ given by composing on the right with g. By the definition of G we have a map of sets $\mathrm{Mor}(GD', C) \to \mathrm{Mor}(GD, C)$. Take $C = G(D')$ and construct an element of $\mathrm{Mor}(GD, GD')$ by taking the image of the identity morphism of $G(D')$ under this map. This gives a morphism $G(g)$, and G is a functor. We call G the left adjoint of F. In particular, $\mathrm{Mor}_{\mathcal{C}}(G(D), C)$ is bijective with $\mathrm{Mor}_{\mathcal{D}}(D, F(C))$ which is reminiscent of the adjoint of a linear map with respect to a bilinear form.

This formulation leads to an alternate definition of adjoint functors.

Definition. *Functors F from a category $\mathcal{C}$ to a category $\mathcal{D}$ and G from $\mathcal{D}$ to $\mathcal{C}$ are adjoints if the two functors from $\mathcal{C} \times \mathcal{D}^{op}$ to the category of sets given by* $\mathrm{Mor}(D, F(C))$ *and* $\mathrm{Mor}(G(D), C)$ *are naturally isomorphic.*

We say that G is the left adjoint of F and that F is the right adjoint of G.

5. Consider the categories $\mathcal{FIELDS}$ of fields and $\mathcal{D}$ of commutative integral domains with only injective ring homomorphisms allowed, with the functor F assigning to a field K the domain K (a type of forgetful functor). Note that ring maps of fields to domains are always injective. The left adjoint G to F (if it exists) would be such that that $\mathrm{Mor}_{\mathcal{D}}(R, F(K))$ is bijective with $\mathrm{Mor}_{\mathcal{FIELDS}}(G(R), K)$ for every field K and commutative domain R. Note that any injective homomorphism of a commutative domain R to a field K extends to the fraction field $FF(R)$ (since all nonzero elements of R are mapped to invertible elements in K), so that this suggests that the left adjoint to the functor F is the fraction field functor FF assigning to every commutative domain its fraction field.

Several of our previous constructions are examples of adjoint functors. Generally constructions involving universal objects are connected to adjoint functors.

6. The forgetful functor F from the category of commutative monoids to the category of abelian groups has the Grothendieck group functor as its left adjoint.

7. The functor which assigns to a ring R its group of invertible elements R^* has a left adjoint given by the group ring functor assigning to a group G its group ring $\mathbf{Z}[G]$.

We end with one reason adjoint functors are useful.

Proposition. *A functor which has a left adjoint preserves limits in a category. A functor which has a right adjoint preserves colimits in a category.*

Proof. We prove the first statement, the other is dual. Let $\mathcal{C}, \mathcal{D}$ be categories and F a functor from $\mathcal{C}$ to $\mathcal{D}$ which has a left adjoint G. Let $\mathcal{J}$ be an index category and T a functor from $\mathcal{J}$ to $\mathcal{C}$. Recall that the limit of this functor is an object $\lim T$ of $\mathcal{C}$ with the property that to give a morphism from an object C to $\lim T$ is the same as giving a morphism from C to $T(J)$ for each object J of $\mathcal{J}$ which is compatible with the morphisms in $\mathcal{J}$. In particular, since F has a left adjoint $\mathrm{Mor}_{\mathcal{D}}(D, F(T(J))$ is bijective with $\mathrm{Mor}_{\mathcal{C}}(G(D), T(J))$. To give compatible morphisms from D to $F(T(J)$ for all J is to give a morphism of D of D to $\lim F \circ T$. On the other hand to give a compatible family of morphisms from $G(D)$ to $T(J)$ for all J is to give a morphism from $G(D)$ to $\lim T$. Using the bijections coming from the adjoint property shows that $F(\lim T)$ is isomorphic to $\lim F \circ T$.

For example, this implies in example 7. that the units of a product of rings $R_1 \times R_2$ is the product of the unit groups $R_1^* \times R_2^*$ since the product of objects is an example of the limit construction. Similarly the free group on a disjoint union of two sets X_1, X_2 is the coproduct of the free group on X_1 with the free group on X_2.

We will see more interesting examples later