Math 551, Assignment 6, due Wednesday, November 20 in class

1. Suppose that the square matrix A is in block diagonal form:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & A_r \end{bmatrix}.$$

Show that $\nu(A) = \sum_{i=1}^r \nu(A_i)$, $\mu_A = \mathrm{lcm}_{i=1}^r \mu_{A_i}$ and $\chi_A = \prod_{i=1}^r \chi_{A_i}$. Here $\nu(A)$ is the nullity (dimension of the nullspace) of A, and μ_A and χ_A are the minimal and characteristic polynomials, respectively. Which of these statements remain true if nonzero entries are allowed anywhere above the diagonal?

2. Let $A \in Mat_{6\times 6}(\mathbf{C})$ and suppose that $(A+I)^{12} = 0$. Show that $(A+I)^6 = 0$. Give a set of representatives for the similarity classes of all such matrices A. Explain briefly how the answer would differ (if at all) over an arbitrary field k instead of $\mathbf{C}$.

3. Let A be a square matrix over a field k. Show that $\mu_A = \mu_{A^T}$ (A^T being the transpose of A). Show that A is similar to A^T. (Hint. First do the case $A = C(f)$ for some monic polynomial f.)

4. Let $H \leq G$, with G and H both free abelian groups of the same finite rank. Let A be the matrix of the inclusion mapping $\alpha : H \to G$ with respect to some bases B and B' of G and H respectively (i.e. the columns of A are the B-coefficients of the elements of H). Show that $|G/H| = \pm \det(A)$.

5. Let G be a finitely generated free abelian group, and let $H \leq G$. Show that there is a unique subgroup $H^* \leq G$ such that (a) $H \leq H^*$; (b) $\mathrm{rank}(H) = \mathrm{rank}(H^*)$; and (c) $G = H^* \oplus K$ for some subgroup K.

6. An elementary matrix is a square matrix which is the same as the identity matrix except possibly in a single off-diagonal entry. Let R be a Euclidean domain. Show that a square matrix $A \in R^{m\times m}$ satisfies $\det(A) = 1$ if and only if A is the product of elementary matrices in $R^{m\times m}$. Deduce that A is invertible in $R^{m\times m}$ if and only if A is the product of elementary matrices and a diagonal matrix whose entries are units in A.

7. Isomorphism-invariant data obtained from all objects in a class (e.g. all abelian groups) are called **invariants** of those objects. A set of invariants is called **complete** when it is true that two objects are isomorphic if and only if they have the same invariants. Which of the following sets of invariants of a finite abelian group G is complete? Give proof or counterexample.

a) $\{|G|$, the exponent of G, the rank of $G\}$ (rank=smallest number of direct summands possible in an expression for G as the direct sum of cyclic groups).

b) $\{n_i \,|\, i \in \mathbf{Z}^+\}$, where for any positive integer i, n_i is the number of elements of G of order i.

8. Let G be a finite subgroup of $GL_n(\mathbf{Q})$, where $\mathbf{Q}$ is the field of rational numbers. Show that G is conjugate (in $GL_n(\mathbf{Q})$) to a subgroup of $GL_n(\mathbf{Z})$. (Hint. Take any $\mathbf{Q}$-basis B of the $\mathbf{Q}$-vector space $\mathbf{Q}^n$ of $n \times 1$ column vectors, and consider the abelian group generated by all gb, $g \in G$, $b \in B$.)