Math 551, Assignment 4, due Friday, October 25 in class

1. A group G is said to be complete if and only if the mapping $G \to \mathrm{Aut}(G)$, $g \mapsto \mathrm{Int}(g)$, is an isomorphism.

a) Show that if H is a normal subgroup of the group X, and if H is complete, then H is a direct factor of X (there exists $C \leq X$ such that $X = H \times C$).

b) For which $n \geq 3$ is A_n complete? What about Σ_n?

2. Subgroups of direct products. Let $G = G_1 \times G_2$, where G_1 and G_2 are groups. Let $\pi_i : G \to G_i$, $i = 1, 2$ be the associated projection mappings. For each subgroup $H \leq G$ define $H_1 = H \cap G_1$ and $H^1 = \pi_1(H)$, and define H_2 and H^2 similarly.

a) Show that $H^1/H_1 \cong H^2/H_2$, and that H has a normal series with quotients isomorphic to H_1, H_2 and H^1/H_1. (cf. Lang's problem entitled "Goursat's Lemma".)

b) Show that if $H \triangleleft G$, then $H_i \triangleleft G$ and $H^i/H_i \leq Z(G/H_i)$, $i = 1, 2$.

c) Show that $A_5 \times A_5$ has $|\,\mathrm{Aut}(A_5)| + 2$ subgroups isomorphic to A_5, and exactly two of them are normal. (Actually, $\mathrm{Aut}(A_5) \cong \Sigma_5$ so there are 122 subgroups.)

3. Let G be a finite group such that for every prime p, G has a unique Sylow p-subgroup G_p. Show that $G = G_{p_1} \times \cdots \times G_{p_r}$ where $p_1, \ldots, p_r$ are the distinct prime divisors of $|G|$.

4. For any abelian groups G and H, the set $\mathrm{Hom}(G, H)$ of homomorphisms from G to H is a group under the operation defined by $(\phi + \phi')(g) = \phi(g) + \phi'(g)$ for all $\phi, \phi' \in \mathrm{Hom}(G, H)$ and all $g \in G$. For any group G define $G^* = \mathrm{Hom}(G, \mathbf{Q}/\mathbf{Z})$ (the "dual" group to G).

a) Show that if G is cyclic, then $G^* \cong G$ or $G^* \cong \mathbf{Q}/\mathbf{Z}$ according as G is finite or infinite.

b) Show that $(G \oplus H)^* \cong G^* \oplus H^*$ for any abelian groups G and H.

c) Show that $G^* \cong G$ for any finite abelian group G.

5. Suppose that F is a free abelian group and B is a basis of F. Let $i : B \to F$ be the inclusion mapping. Verify the universal mapping property: for every abelian group G and mapping (of sets) $j : B \to G$, there exists a unique homomorphism $\phi : G \to H$ such that the diagram commutes:

$$\begin{tikzcd} B \arrow[r] \arrow[dr] & F \arrow[d] \\ & G \end{tikzcd}$$

6. a) Let G be a free abelian group and $x \in G$. Show that x is an element of some basis of G if and only if there exists a homomorphism $\phi : G \to \mathbf{Z}$ such that $\phi(x) = 1$.

b) Let $n > 1$ be an integer, and let $G = \mathbf{Z}^n$ be the (free abelian) group of all integer $1\times n$ row vectors, under vector addition. Let $x = [x_1\, x_2\, \cdots\, x_n]$ be a nonzero element of G. Show that the greatest common divisor of $x_1, x_2, \ldots, x_n$ (ignoring any x_i's which are 0) is 1 if and only if there exists an integer $n \times n$ matrix A whose first row is x and such that $\det A = 1$. You may use familiar properties of determinants. (Hint for "only if": show that x is an element of some basis of G.)

c) For any abelian group G and integer n define $nG = \{ng \mid g \in G\}$. A subgroup H of G is said to be pure in G if and only for every integer n, we have $nG \cap H = nH$. Show that if G is a free abelian group of finite rank and $H \leq G$, then H is a direct summand of G if and only if H is pure in G.

7. Show that $G := \mathbf{Z}_{p^3} \oplus \mathbf{Z}_p$ has a subgroup $H \cong \mathbf{Z}_{p^2}$ such that $G/H \cong \mathbf{Z}_{p^2}$.

Additional questions, not part of the assignment.

A. Show that direct summands are always pure (see 6c). On the other hand, let p be a prime and set $G = \prod_{n=1}^{\infty} \mathbf{Z}_{p^n}$ and $H = \coprod_{n=1}^{\infty} \mathbf{Z}_{p^n} \leq G$. Let $T(G) = \{g \in G \mid g \text{ has finite order}\}$. Show that H is a pure subgroup of $T(G)$ but is not a direct summand of $T(G)$. (Hint for second part. Show that $T(G)/H = p^n(T(G)/H)$ for all n, but the only subgroup C of G such that $C = p^n C$ for all n is $C = 0$.)

B. After his Jacqueline Lewis Lecture on October 17, Richard Stanley posed the following (open?) question. If the answer is "yes", it would give another proof of the "saturation conjecture". To save space use the notation

$$[[m_1, m_2, \ldots, m_r]] := \mathbf{Z}_{m_1} \oplus \mathbf{Z}_{m_2} \oplus \cdots \oplus \mathbf{Z}_{m_r}$$

for any positive integers $m_1, \ldots, m_r$. Let p be a prime, and $q = p^m$ where m is a positive integer. Let r be a positive integer and $\lambda_1, \ldots, \lambda_r$, $\mu_1, \ldots, \mu_r$, $\nu_1, \ldots, \nu_r$ all be nonnegative integers. Suppose that

$$G := [[q^{\lambda_1}, \ldots, q^{\lambda_r}]] \text{ has a subgroup } H \text{ such that}$$
$$H \cong [[q^{\mu_1}, \ldots, q^{\mu_r}]] \text{ and } G/H \cong [[q^{\nu_1}, \ldots, q^{\nu_r}]].$$

Does it follow that

$$G_0 := [[p^{\lambda_1}, \ldots, p^{\lambda_r}]] \text{ has a subgroup } H_0 \text{ such that}$$
$$H_0 \cong [[p^{\mu_1}, \ldots, p^{\mu_r}]] \text{ and } G_0/H_0 \cong [[p^{\nu_1}, \ldots, p^{\nu_r}]]\,?$$