Math 551, Assignment 3, due Wednesday, October 16 in class

As usual, G is a group.

1. Prove the following two statements using group actions.

a) Suppose that $H \leq G$ and $|G : H| = n < \infty$. Show that there is a subgroup $K \leq H$ such that $K \triangleleft G$ and $|G : K|$ divides $n!$, and in particular $|G : K| < \infty$.

b) Suppose that $H_1 \leq G$ and $H_2 \leq G$ are two subgroups of finite index in G. Show that $H_1 \cap H_2$ has finite index in G. (Hint. Use an action on the disjoint union $(G/H_1) \cup (G/H_2)$.)

2. Let Ω be an infinite set and define

$$\Sigma_\Omega^o = \{\sigma \in \Sigma_\Omega \,|\, \sigma(\omega) = \omega \text{ for all but finitely many } \omega \in \Omega\}.$$

a) Show that $\Sigma_\Omega^o \triangleleft \Sigma_\Omega$.

b) Make a reasonable definition of A_Ω^o, and show that A_Ω^o is simple.

3. Suppose that $G = \langle S \rangle$, and let X be a subset of G. Show that $X = G$ if and only if $(S \cup S^{-1})X \subseteq X$, where $S^{-1} = \{s^{-1} \,|\, s \in S\}$.

4. Let $G = gp\langle s_1, \ldots, s_n \,|R\rangle$, where R is the set of all relators of the form

$$s_i^2,\ 1 \leq i \leq n$$
$$(s_i s_j)^2,\ 1 \leq i < j \leq n,\ j > i+1$$
$$(s_i s_{i+1})^3,\ 1 \leq i < n$$

Show that $G \cong \Sigma_{n+1}$. (Hints. Map $s_i \mapsto (i\ i+1) \in \Sigma_{n+1}$. Also, let H be the subgroup of G generated by $s_1, \ldots, s_{n-1}$, and show that $G = H \cup s_n H \cup s_{n-1} s_n H \cup \cdots \cup s_1 s_2 \cdots s_n H$.)

5. Each of the following statements is false. Give a counterexample in each case. Also alter the underlined part of the statement in as reasonable way as possible so that the statement becomes true; then prove the altered statement.

a) If $\phi : G \to H$ is any homomorphism and L and M are <u>any subgroups of G</u>, then $\phi(L \cap M) = \phi(L) \cap \phi(M)$.

b) If K char G and $K \leq L \leq G$, then L char G <u>if and only if</u> L/K char G/K.

c) If $P \in \mathrm{Syl}_p(G)$ and H is <u>any subgroup of G</u>, then $P \cap H \in \mathrm{Syl}_p(H)$.

d) If G possesses a composition series and H is <u>any subgroup of G</u>, then H possesses a composition series.

e) If A, B and C are subgroups of G such that $A \triangleleft B$, then $A \cap C \triangleleft B \cap C$ and <u>$B \cap C/A \cap C \cong B/A$</u>.

6. If k is any field and n is any natural number, then we define $Z = \{cI \,|\, c \in k^\times\}$, $PGL_n(k) = GL_n(k)/Z$ and $PSL_n(k) = SL_n(k)/Z \cap SL_n(k)$. Show that $PGL_n(k)$ has a normal subgroup $H \cong PSL_n(k)$, and that $PGL_n(k)/H \cong k^\times/(k^\times)^n$.

7. Suppose that G is a finite solvable group. Show that if $x_1, \ldots, x_r \in G$ are elements of orders $p_1^{a_1}, \ldots, p_r^{a_r}$, respectively, where $p_1, \ldots, p_r$ are *distinct* primes and $a_1, \ldots, a_r$ are nonnegative integers, and if $x_1 x_2 \cdots x_r = 1$, then $a_1 = a_2 = \cdots = a_r = 0$. (Hint. Use induction. Remark. It is also true that any finite group with this property is solvable, but this is a very deep theorem.)

8. Show that up to isomorphism, there are exactly three nonabelian groups of order 12: A_4, $Z_2 \times \Sigma_3$ and $gp\langle x, y \,|\, x^4, y^3, xyx^{-1}y\rangle$.