Math 551, Assignment 2, due Friday, September 27 in class

1. Let G act on Ω, $\omega \in \Omega$ and $H \leq G$. Show that H acts transitively on Ω if and only if G acts transitively on Ω and $G = G_\omega H$.

2. (The "Frattini Argument") Let N be a finite normal subgroup of G, and let $P \in \mathrm{Syl}_p(N)$ for some prime p. Show that $G = N_G(P)N$.

3. Let G be a finite group, $G \neq 1$. Define $\Phi(G)$ (the "Frattini subgroup" of G) to be the intersection of all the maximal subgroups of G. Show that $\Phi(G) \triangleleft G$ and that every Sylow p-subgroup of $\Phi(G)$ for every prime p is also normal in G.

4. Let $H \leq G$ and set $K = \bigcap_{g \in G} {}^gH$. Show that $K \triangleleft G$ and that any normal subgroup of G lying in H actually lies in K.

5. A finite group G is called **nilpotent** if and only if for each subgroup $H \leq G$ such that $H \neq G$, it is the case that $H \neq N_G(H)$. Show that any group of prime power order p^a is nilpotent. (Hint. Consider the conjugation action of H on the set Ω of all G-conjugates of H.)

6. Let $\sigma = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $\tau = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $Q_8 = \langle \sigma, \tau \rangle \leq SL_2(\mathbf{C})$, the "quaternion" group. Show that

a) Q is a non-abelian group of order 8 in which every subgroup is normal

b) If H is any group such that $H = \langle s, t \rangle$, $s^4 = 1$, $t^2 = s^2$ and $tst^{-1} = s^{-1}$, then there is a surjective homomorphism $Q \to H$

c) There are exactly two isomorphism classes of nonabelian groups of order 8, namely those represented by D_8 and Q. (Hint. By a previous exercise, if $|G| = 8$ and $K \leq G$ with $|K| = 4$, then $K \triangleleft G$.)

7. Show that any group of order 56 has a normal Sylow 2-subgroup, and that Σ_7 has no subgroup of order 56. (Hint for second part: If $H \leq \Sigma_7$ with $|H| = 56$, then in the action of H on 7 letters, what is the stabilizer of a point?)

8. Let $G = GL_3(\mathbf{Z}/2\mathbf{Z})$. Show that $|G| = 168$, and that the upper triangular subgroup T is a Sylow 2-subgroup of G. Show that every element of G of order 2 is conjugate to

$$t = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Show that $C_G(t) = T$, and that G has 21 elements of order 2, 42 elements of order 4, 48 elements of order 7, and (for extra credit) 56 elements of order 3.

9. Let G be a finite group and let p be a prime divisor of $|G|$. Let $\mathcal{S}(p) = \{H \mid H \leq G \text{ and } |H| = p\}$. Show that $|\mathcal{S}(p)| \equiv 1 \bmod p$. (Hint. Let a Sylow p-subgroup act by conjugation.)

10. Let $n \geq 4$ and $G = \Sigma_n$. Begin by taking an arbitrary element $v \in G$ of order 2 and computing $|C_G(v)|$ in terms of the number of disjoint 2-cycles in the cycle decomposition of v.

a) Show that if $t = (ab) \in G$ is a transposition and $u \in G$ is any element of order 2 such that $|C_G(t)| = |C_G(u)|$, then u is also a transposition, unless $n = 6$.

b) Show that if $\alpha \in \mathrm{Aut}(G)$ and $t \in G$ is a transposition, then $\alpha(t)$ is a transposition.

c) Show that if $t = (ab)$ and $u = (cd)$ are transpositions, then $\{a, b\} \cap \{c, d\}$ consists of a single point if and only if $|tu| = 3$.

d) Using b) and c), show that if $n \neq 6$, then $\mathrm{Aut}(G) = \mathrm{Inn}(G) \cong G$.

e) Show that if $n = 6$, then $\mathrm{Aut}(G) \neq \mathrm{Inn}(G)$, by doing the appropriate exercise in Lang, at the end of chapter 1.