Math 551 – Algebra – Fall 2000

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C. Categories

Read the masters: *Categories for the Working Mathematician*, by Saunders Mac Lane, is an invaluable reference.

1. Definitions and examples.

1a. Definition

Definition. *A category* $\mathbf{C}$ *consists of*

a) *a class* $\mathcal{O}bj(\mathbf{C})$ *(whose elements are called the "objects of* $\mathbf{C}$*");*

b) *for each* $A, B \in \mathcal{O}bj(\mathbf{C})$*, a set* $\mathrm{Mor}_{\mathbf{C}}(A, B)$*, or more simply* $\mathrm{Mor}(A, B)$ *(whose elements are called the "morphisms from* A *to* B *(in* $\mathbf{C}$*)");*

c) *for each* $A, B, C \in \mathcal{O}bj(\mathbf{C})$*, a function* $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$ *(written* $(f, g) \mapsto g \circ f$ *and called "composition"), such that*

   1) *composition is associative, that is: for each* $A, B, C, D \in \mathcal{O}bj(\mathbf{C})$ *and each* $f \in \mathrm{Mor}(A, B)$*,* $g \in \mathrm{Mor}(B, C)$*, and* $h \in \mathrm{Mor}(C, D)$*,* $h \circ (g \circ f) = (h \circ g) \circ f$*; and*

   2) *every object has an identity morphism, that is: for each* $A \in \mathcal{O}bj(\mathbf{C})$*, there is* $1_A \in \mathrm{Mor}(A, A)$ *such that for any* $B \in \mathcal{O}bj(\mathbf{C})$*, any* $f \in \mathrm{Mor}(A, B)$ *and any* $g \in \mathrm{Mor}(B, A)$*, we have* $f \circ 1_A = f$ *and* $1_A \circ g = g$*.*

1b. Examples

Ex. 1. The category $\mathbf{Gp}$ of groups. Here the objects are all groups, i.e., $\mathcal{O}bj(\mathbf{Gp})$ is the class of all groups. For any groups G, H, $\mathrm{Mor}(G, H)$ is the set of all homomorphisms from G to H. Composition $\circ$ is defined just to be the usual composition of mappings (it is necessary to know that the composite of homomorphisms is a homomorphism, so that the composite ends up in $\mathrm{Mor}(A, C)$!). 1_G is the identity mapping $G \to G$. The various parts of the definition are easily checked.

Ex. 2. Similarly there are the categories

a) **Set** (objects: all sets; morphisms from A to B: all functions $A \to B$)

b) **Ab** (objects: all abelian groups; morphisms from A to B: all group homomorphisms $A \to B$)

c) ${}_{\mathbf{R}}\mathbf{Mod}$, for a fixed ring R (objects: all left R-modules; morphisms: all R-module homomorphisms)

d) **Top** (objects: all topological spaces; morphisms: all continuous mappings)

e) **Diff** (objects: all differentiable (real) manifolds; morphisms: all differentiable mappings)

f) etc., etc.

Ex. 3. All the preceding examples are "concrete" categories; that is, the objects are (certain) sets, and the elements of $\mathrm{Mor}(A, B)$ are (certain) functions from A to B. Although many categories are "concrete", this is by no means always the case. Nevertheless, we formally write an element $\alpha \in \mathrm{Mor}(A, B)$ as an arrow

$$\alpha : A \to B.$$

As an example of a non-concrete category, let **Delta** be the category with three objects, A, B and C, and with only the following morphisms: the three identity morphisms and the following three others: $\alpha : A \to B$, $\beta : B \to C$ and $\gamma : A \to C$. (Thus $\mathrm{Mor}(B, A)$ is empty, for instance.) The composition mappings are forced by the axioms; the only nontrivial composite is $\beta \circ \alpha = \gamma$. This is an example of a "diagram" category. Another common diagram category is the category with two objects A, B, and precisely two morphisms from A to B, and none from B to A.

Ex. 4. On the more concrete level there are still more subtle ways to form categories. For instance **PTop** is the category of all "pointed topological spaces"; its objects are ordered pairs (X, x) consisting of a nonempty topological space X and a point $x \in X$; a morphism $(X, x) \to (Y, y)$ is by definition a continuous function $\phi : X \to Y$ such that $\phi(x) = y$.

Ex. 5. The definition of category is "symmetric"; we could form a (somewhat bizarre) category whose objects are groups, but such that $\mathrm{Mor}(A, B)$ consists of all morphisms from B to A. (!) The composition law would then be: given $\alpha \in \mathrm{Mor}(A, B)$ and $\beta \in \mathrm{Mor}(B, C)$, $\beta \circ \alpha$ would be the composite (β followed by α), a homomorphism from C to A and hence an element of $\mathrm{Mor}(C, A)$.

More generally, given any category $\mathbf{C}$, there is a dual category $\mathbf{C}'$, given by "turning the arrows around": $\mathcal{O}bj(\mathbf{C}') = \mathcal{O}bj(\mathbf{C})$, and $\mathrm{Mor}_{\mathbf{C}'}(A, B) = \mathrm{Mor}_{\mathbf{C}}(B, A)$; the identity morphisms are the same in $\mathbf{C}$ as in $\mathbf{C}'$, and composition in $\mathbf{C}'$ is the "opposite": $\alpha \circ_{\mathbf{C}} \beta = \beta \circ_{\mathbf{C}'} \alpha$.

Duality Principle. *Any theorem holding for an arbitrary category holds with all arrows reversed.*

Namely, the "dual" statement in $\mathbf{C}$ is equivalent to the original statement in $\mathbf{C}'$.

2. Some categorical concepts.

Definition. *A morphism $\alpha : A \to B$ in a category $\mathbf{C}$ is an isomorphism in $\mathbf{C}$ if and only if there exists $\beta : B \to A$, also a morphism in $\mathbf{C}$, such that $\beta \circ \alpha = 1_A$ and $\alpha \circ \beta = 1_B$.*

Exercise. *1_A is an isomorphism. The composite of isomorphisms (if defined) is an isomorphism.*

2a. Initial and terminal objects

Definition. *Let $\mathbf{C}$ be a category. An object $A \in \mathcal{O}bj(\mathbf{C})$ is an initial object in $\mathbf{C}$ if and only if for every $B \in \mathcal{O}bj(\mathbf{C})$, there is a unique morphism $A \to B$ in $\mathbf{C}$.*

The unique morphism $A \to A$ has to be 1_A, of course.

Ex. 1. In **Set**, the empty set $\emptyset$ is an initial object. It is actually the only initial object. In **Gp**, the trivial group $\{1\}$ is an initial object. Any group with one element is an initial object. Not every category has an initial object, however. For instance the diagram category $\cdot \rightrightarrows \cdot$ has no initial object.

Ex. 2. Universal mapping properties can be formulated in the language of initial objects. For example, let S be a set and let $\mathbf{C}$ be the category in which an object is a mapping

$$\phi : S \to G$$

from S into some group G (i.e. an object is a pair (G, ϕ) consisting of a group G and a mapping $\phi : S \to G$); and given two objects

$$\phi : S \to G, \quad \phi' : S \to G'$$

a morphism from the first to the second is defined to be a group homomorphism $\psi : G \to G'$ making the diagram commute:

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & G \arrow[d] \\ & G' \end{tikzcd}$$

Composition is just ordinary composition of mappings. To be sure, something needs to be checked: given two morphisms, i.e., two commutative diagrams like the one above, if the morphisms can be composed (i.e. the diagrams fit together) then the composite diagram commutes. This is easy to check.

Notice also that a morphism in $\mathbf{C}$ is an isomorphism if and only if it is just an isomorphism of groups (in addition to the having the diagram-commuting property required). This is because if ϕ is an isomorphism in the above diagram, then ϕ^{-1} also makes the diagram commute (check it!).

In this language, a free group on S is just an initial object in $\mathbf{C}$.

One benefit of this abstraction here is that the uniqueness of free groups becomes just an instance of the following trivial theorem:

Theorem. *If A and A' are two initial objects in the category $\mathbf{C}$, then there exists a unique isomorphism from A to A'.*

Proof. There exist unique morphisms $\alpha : A \to A'$ and $\beta : A' to A$. Then $\alpha \circ \beta$ is a morphism from $A' \to A'$, but since A' is initial the only such morphism is $1_{A'}$. Therefore $\alpha \circ \beta = 1_{A'}$, and similarly $\beta \circ \alpha = 1_A$.

The dual notion is that of terminal object:

Definition. *If A is an object in $\mathbf{C}$, then A is terminal in $\mathbf{C}$ if and only if for each $B \in \mathcal{O}bj(\mathbf{C})$, there is a unique morphism $B \to A$.*

Equivalently: A is initial in $\mathbf{C}'$. The dual theorem to the above theorem simply replaces the word "initial" by the word "terminal".

The theorem and its dual imply that the solution to any appropriately formulated universal mapping problem is unique (if it exists), up to a unique isomorphism.

In **Set**, any 1-element set is a terminal object. In **Gp** or **Ab** or $_{\mathbf{R}}$**Mod**, $\{1\}$, 0 and 0 are terminal objects. In the category of commutative rings with identity element $1 \neq 0$, (and homomorphisms preserving taking 1 to 1), there is no terminal object. In fact there is no commutative ring to which you can map both $\mathbf{Z}_2$ and $\mathbf{Z}_3$. (If you could, it would force $1 + 1 = 1 + 1 + 1 = 0$, so $1 = 0$.) However, if the 0 ring is permitted as an object, then it is of course a terminal object. In both these categories of rings, $\mathbf{Z}$ is an initial object.

In **PTop**, a one-point topological space is both initial and terminal.

2b. Products and coproducts

Definition. *If $\{A_i\}_{i \in I}$ is a family of objects in the category $\mathbf{C}$, then a product of this family in $\mathbf{C}$ is an object A together with morphisms $\pi_i : A \to A_i$, one for each $i \in I$, satisfying the following universal property: for every object B in $\mathbf{C}$ and every family $\{\beta_i\}_{i \in I}$ of morphisms $\beta_i : B \to A_i$, there is a unique morphism $\beta : B \to A$ such that the following diagrams commute for each $i \in I$:*

$$\begin{array}{ccc} B & \rightarrow & A \\ & \searrow & \downarrow \\ & & A_i \end{array}$$

A coproduct of $\{A_i\}$ in $\mathbf{C}$ is a product of $\{A_i\}$ in $\mathbf{C}'$.

A product is just a terminal object in a suitable category, so by the theorem of the previous section, products are uniquely determined, up to a unique isomorphism.

Examples. In **Set**, Cartesian products are products in this categorical sense. In **Gp** and **Ab**, products exist as well: they are given by the direct product of groups. The same is the case for the category of rings with identity, rings with(out) identity ("rngs"), commutative rings with identity, topological spaces, pointed topological spaces.

Coproducts are more diverse. In **Set**, *or* **Top** the coproduct of $\{A_i\}$ is the disjoint union of the A_i. In **Gp**, the free product of $\{G_i\}$ is a coproduct. In **Ab** or ${}_{\mathbf{R}}\mathbf{Mod}$, the direct sum of $\{A_i\}$ is their coproduct.

Exercise. What are coproducts in **PTop**?

Exercise. Show that in the category of rings with identity (including the 0 ring as an object), 0 is a coproduct of $\mathbf{Z}_2$ and $\mathbf{Z}_3$. Is there a coproduct of $\mathbf{Z}$ and $\mathbf{Z}_2$?

Exercise. If A is an initial object, then any object B is a coproduct of A and B (with respect to morphisms which you should specify).

2c. Limits and colimits

Given any commutative diagram Δ (of any shape whatsoever!) in a category $\mathbf{C}$, consisting of objects $A_i, i \in I$ and various morphisms, we can consider the category $\mathbf{C}_\Delta$ defined as follows. An object in $\mathbf{C}_\Delta$ is an object B of $\mathbf{C}$ together with morphisms $\phi_i : B \to A_i$, $i \in I$, forming with Δ a commutative diagram. If B' and ϕ'_i constitute another object, a morphism $B \to B'$ in $\mathbf{C}_\Delta$ is just a morphism $\psi : B \to B'$ in $\mathbf{C}$ such that $\phi'_i \psi = \phi_i$ for all $i \in I$. A terminal object in $\mathbf{C}_\Delta$ is called a limit of Δ in $\mathbf{C}$. As usual, this is unique up to unique isomorphism.

The dual notion is that of colimit.

For example, in the category of sets, if we have an ascending sequence of injections: $A_0 \to A_1 \to \cdots \to A_n \to \cdots$, then its colimit is the union of the A_i, when identification is made of A_i with its image in A_{i+1}.

On the other hand if we have a descending sequence of surjections $\phi_i : A_i \to A_{i-1}$, say $\cdots \to A_n \to \cdots \to A_1 \to A_0$, then its limit mady be constructed as a subset of the Cartesian product $\prod_i A_i$ consisting of all tuples $(a_i)_{i \in I}$ which are "coherent" in the sense that $\phi_i(a_i) = a_{i-1}$ for all $i \geq 1$.

If there are no morphisms among the A_i, then their limit is their product and their colimit is their coproduct (if existent).

The limit of a diagram $A \to B \leftarrow C$ is called a pull-back, and the colimit of a diagram $A \leftarrow B \to C$ is called a push-out.

In the category of sets or abelian groups or R-modules or topological spaces, the pullback of $A \to B \leftarrow C$ can be constructed as the subset (subgroup, submodule, subspace) of the product $A \times C$ consisting of all pairs (a, c) such that the images of a and c in B coincide. Likewise the pushout of $A \leftarrow B \to C$ can be constructed as a quotient of the coproduct of A and C, by the smallest equivalence relation $\sim$ such that for any $b \in B$, its images in A and C are equivalent (when mapped into the coproduct).

Limits and colimits are sometimes called inverse limits and direct limits.

Limits and colimits do not necessarily exist in arbitrary categories, but they are unique when they exist.

2d. Monics and Epics

Definition. *A morphism $\phi : A \to A'$ in a category* **C** *is monic if and only if whenever $\psi_1, \psi_2 : B \to A$ are two morphisms and $\phi \circ \psi_1 = \phi \circ \psi_2$, then $\psi_1 = \psi_2$.*

The dual notion is “epic”.

In **Set**, monic is equivalent to injective, and epic is equivalent to surjective. The same holds in **Ab** and in $_{\mathbf{R}}\mathbf{Mod}$, as well as in **Gp**; but this is tricky to prove for epics in **Gp**. What is needed is to construct, given a proper subgroup H of a group G, two homomorphisms from G to some group K which are distinct but agree on H. (Hint. Try $K = \Sigma_\Omega$, where Ω is the union of G/H with one more point.)

3. Functors.

3a. Definition and Examples

A functor is a “morphism of categories”. That is, given two categories $\mathbf{C}$ and $\mathbf{C}'$, a functor $F : \mathbf{C} \to \mathbf{C}'$ associates to each object $A \in \mathcal{O}bj(\mathbf{C})$ an object $FA \in \mathcal{O}bj(\mathbf{C}')$, and also associates to each morphism $\alpha : A \to B$ in $\mathbf{C}$ a morphism $F\alpha : FA \to FB$ in $\mathbf{C}'$, such that $F(1_A) = 1_{FA}$ for each $A \in \mathcal{O}bj(\mathbf{C})$, and $F(g \circ h) = Fg \circ Fh$ whenever the left side is defined.

Examples.

1. $U : \mathbf{Gp} \to \mathbf{Set}$. Here $UG = G$ (ignoring the operation) and $Uf = f$ (just as a mapping of sets). U is a “forgetful” functor. There are lots of forgetful functors: ${}_{\mathbf{R}}\mathbf{Mod} \to \mathbf{Ab}$, $\mathbf{PTop} \to \mathbf{Top}$, $\mathbf{Ab} \to \mathbf{Set}$, etc., etc.

2. Let $\mathbf{C}$ be a category and A a fixed object in C. Then $F = \mathrm{Mor}(A, \cdot)$ is a functor from $\mathbf{C}$ to $\mathbf{Set}$. Specifically, this functor takes the object B to the set $\mathrm{Mor}(A, B)$, and takes the morphism $\phi : B \to C$ to the function $\mathrm{Mor}(A, B) \to \mathrm{Mor}(A, C)$ which is composition with ϕ. The verification of the axiom $F(f \circ g) = F(f) \circ F(g)$ (where $g : B \to C$ and $f : C \to D$) is that for any $h \in \mathrm{Mor}(A, B)$, $(f \circ g) \circ h = f \circ (g \circ h)$.

3. Let $\mathbf{C}$ be a category and A a fixed object in $\mathbf{C}$. Then $\mathrm{Mor}(\cdot, A)$ is a functor from the opposite category $\mathbf{C}'$ to $\mathbf{Set}$, with a definition similar to the previous. This time, if $\phi : B \to C$, we get $F(\phi) : \mathrm{Mor}(A, C) \to \mathrm{Mor}(A, B)$ by $f \mapsto f \circ \phi$. Moreover, for $g : B \to C$ and $f : C \to D$, we have $F(f \circ g)(h) = h \circ (f \circ g) = (h \circ f) \circ g = F(g)(F(f)(h))$ for an arbitrary $h \in \mathrm{Mor}(A, D)$. A functor on $\mathbf{C}'$ is usually called a “contravariant functor on $\mathbf{C}$”.

4. If $\mathbf{C}$ is a category, then $\mathrm{Mor}(\cdot, \cdot)$ is a functor from $\mathbf{C}' \times \mathbf{C}$ to $\mathbf{Set}$.

5. Let F be a field and ${}_F MOD'$ the category of F-vector spaces. Then duality gives a functor F such that $F(V) = V^*$ for each V, and $F(T) = T^*$. This is a contravariant functor. Its “square” is the covariant functor $G(V) = V^{**}$, $G(T) = T^{**}$.

6. The “fundamental group functor” $\pi_1 : \mathbf{PTop} \to \mathbf{Gp}$ associates to each pointed topological space X its fundamental group $\pi(X)$, whose elements are the homotopy classes of loops in X (starting and ending at the base point); the group operation is to follow one loop by the other; the inverse of a loop is the same loop in the opposite direction. A continuous mapping $X \to Y$ induces a mapping $\pi_1(X) \to \pi_1(Y)$ in the obvious way which is a homomorphism of groups, and the functor axioms are easily checked.

7. Similarly there are higher homotopy group functors, and also homology group functors.

8. Let Δ be the category with three objects, and just three non-identity morphisms, namely

$$\begin{tikzcd} \cdot \arrow[r] \arrow[dr] & \cdot \arrow[d] \\ & \cdot \end{tikzcd}$$

A functor from Δ to a category $\mathbf{C}$ is just a commutative triangle in $\mathbf{C}$. In this way commutative diagrams of various shapes in a category $\mathbf{C}$ may be thought of as functors to

$\mathbf{C}$ from a "diagram" category such as Δ.

9. The "free group functor" $F : \mathbf{Set} \to \mathbf{Gp}$ associates to each set X the free group FX on X, which comes equipped with a mapping $i_X : X \to FX$. (There are many free groups, all isomorphic; we pick one for each X.) For every set morphism $f : X \to Y$, there exists a unique homomorphism $FX \to FY$ such that the following diagram commutes, and we call that morphism Ff:

$$\begin{array}{rcl} i_X : X & \longrightarrow & FX \\ f \downarrow & & Ff \downarrow \\ i_Y : Y & \longrightarrow & FY \end{array}$$

The uniqueness of Ff is vital for verifying that $F(f \circ g) = Ff \circ Fg$ and $F(1_X) = 1_{FX}$.

In a similar way there exist free module and free abelian group functors, from $\mathbf{Set}$ to the appropriate category.

Exercise. *Functors carry isomorphisms to isomorphisms.*

3b. Natural Transformations

Let F and G be functors from $\mathbf{C}$ to $\mathcal{D}$. A natural transformation $\Theta : F \to G$ is a family of morphisms in $\mathcal{D}$,

$$\Theta_A : FA \to GA,$$

one for each object in $\mathbf{C}$, such that for every morphism $f : A \to B$ in $\mathbf{C}$, the diagram

$$\begin{array}{rcl} F(f) : FA & \longrightarrow & FB \\ \Theta_A \downarrow & & \Theta_B \downarrow \\ G(f) : GA & \longrightarrow & GB \end{array}$$

commutes.

Thus for example our mapping $\nu_V : V \to V^{**}$ for finite-dimensional vector spaces is a natural transformation from the identity functor to the double-dual functor.

A topological example: the Hurewicz homomorphism $\pi_1(X) \to H_1(X)$ is a natural transformation from the π_1-functor to the H_1-functor.

The mapping i_X from a set X to the free group FX defines a natural transformation $i : 1_{\mathbf{Set}} \to F$. Indeed the appropriate diagram was part of our definition of F in the last section.

A natural transformation Θ is a natural isomorphism if every Θ_A is an isomorphism.

The isomorphism $A \oplus B \cong B \oplus A$, $(a, b) \mapsto (b, a)$, is a natural isomorphism of abelian groups. Translation: Let $F : \mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$ be the functor such that $F(A, B) = A \oplus B$, and $F(f, g) = f \oplus g$, where for $f : A \to A'$ and $g : B \to B'$ we define $f \oplus g : A \oplus B \to A' \oplus B'$ by $(f \oplus g)(a, b) = (f(a), g(b))$. (Check that F is a functor!) Let $G : \mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$ be the functor defined by $G(A, B) = B \oplus A$ and $G(f, g) = g \oplus f$. Then the mappings $\Theta_{(A,B)} : A \oplus B \to B \oplus A$, $(a, b) \mapsto (b, a)$, are isomorphisms and constitute a natural transformation from F to G.

Exercise. *Show that if R is a ring and M is a left R-module, then $\mathrm{Hom}_R({}_R R, M) \cong M$ (as sets), via a natural isomorphism. Show that if R is commutative, then this is a natural isomorphism of R-modules. (The main point of this exercise is to make you formulate precisely what the assertion you are asked to prove; then proving it should be easy.)*

Natural transformations have to be between functors with the same domain and codomain (range). Thus it is impossible to have a natural transformtion from the vector space V to its dual V^*, since the functor $FV = V$ is defined on the category of vector spaces over a given field, while $GV = V^*$ is defined on the opposite category.

Exercise. *Two categories $\mathbf{C}$ and $\mathbf{C}'$ are called equivalent if and only if there exist functors $F : \mathbf{C} \to \mathbf{C}'$ and $G : \mathbf{C}' \to \mathbf{C}$ such that $1_{\mathbf{C}}$ and $G \circ F$ are naturally isomorphic, and $1_{\mathbf{C}'}$ and $F \circ G$ are naturally isomorphic. Explain why this is a more reasonable notion than "isomorphism" of categories. Let $\mathbf{C}$ be the category of all rational vector spaces, $\mathbf{C}'$ the category of all real vector spaces, and $\mathbf{C}''$ the "sub"category of $\mathbf{C}$ consisting of the spaces 0, $\mathbf{Q}$, $\mathbf{Q} \oplus \mathbf{Q}$, etc., one for each dimension (and all morphisms between them). Show that $\mathbf{C}$, $\mathbf{C}'$ and $\mathbf{C}''$ are all equivalent.*

Exercise. *Let $\mathbf{C}$ and $\mathcal{D}$ be categories. Show that there is a category $\mathrm{Mor}(\mathbf{C}, \mathcal{D})$ whose objects are the functors from $\mathbf{C}$ to $\mathcal{D}$ and whose morphisms are the natural transformations.*

Exercise. *In the previous exercise, if $\mathbf{C}$ is the "diagram" category*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[dr] & \cdot \arrow[d] \\ & \cdot \end{tikzcd}$$

and $\mathcal{D} = \mathbf{Ab}$, describe $\mathrm{Mor}(\mathbf{C}, \mathcal{D})$.

Exercise. *(Yoneda's Lemma) Let $F : \mathbf{C} \to \mathbf{Set}$ be a functor. Then for any object A of $\mathbf{C}$ there is a bijection*

$$Nat(\mathrm{Mor}(A, \cdot), F) \cong FA, \ \Theta \mapsto \Theta_A(1_A).$$

Here $Nat(\mathrm{Mor}(A, \cdot), F)$ is the set of all natural transformations from $\mathrm{Mor}(A, \cdot)$ to F.

3c. Adjoint Functors

If $F : \mathbf{C} \to \mathcal{D}$ is a functor, then $\mathrm{Mor}(F(\cdot), \cdot)$ is a functor from $\mathbf{C}' \times \mathcal{D} \to \mathbf{Set}$. Likewise if $G : \mathcal{D} \to \mathbf{C}$ is a functor, then $\mathrm{Mor}(\cdot, G(\cdot)) : \mathbf{C}' \times \mathcal{D} \to \mathbf{Set}$ is a functor. The functors F and G are called **adjoint** (more precisely, F is left adjoint to G, and G is right adjoint to F) if and only if there is a natural isomorphism between these two functors on $\mathbf{C}' \times \mathcal{D}$, i.e., there is a natural isomorphism

$$\Theta_{A,B} : \underset{\mathbf{C}}{\mathrm{Mor}}(FA, B) \cong \underset{\mathcal{D}}{\mathrm{Mor}}(A, GB).$$

The terminology seems to be derived from the formal similarity with the definition of the adjoint of a linear operator. Whatever the terminology, this notion is quite powerful.

The naturality means that for any morphisms $f : A' \to A$ in $\mathbf{C}$ and $g : B \to B'$ in $\mathcal{D}$, the diagram

$$\begin{array}{rccc} \mathrm{Mor}(Ff, g): & \mathrm{Mor}(FA, B) & \longrightarrow & \mathrm{Mor}(FA', B') \\ & \Theta_{A,B} \downarrow & & \Theta_{A',B'} \downarrow \\ \mathrm{Mor}(f, Ug): & \mathrm{Mor}(A, UB) & \longrightarrow & \mathrm{Mor}(A', UB') \end{array}$$

commutes.

Lemma. *For this naturality to hold, it is necessary and sufficient that it hold whenever $g = 1_B$ and also whenever $f = 1_A$.*

This is because we can insert an extra column for the pair (A', B), and $\mathrm{Mor}(Ff, g)$ is the composite of $\mathrm{Mor}(Ff, 1_B)$ and $\mathrm{Mor}(1_{FA'}, g)$, and similarly $\mathrm{Mor}(f, Ug)$ is a composite.

Examples.

1. The free functor $F : \mathbf{Gp} \to \mathbf{Set}$ is left adjoint to the forgetful functor $U : \mathbf{Set} \to \mathbf{Gp}$. That is, for any set X and group G,

$$\mathop{\mathrm{Mor}}_{\mathbf{Gp}}(FX, G) \cong \mathop{\mathrm{Mor}}_{\mathbf{Set}}(X, UG),$$

naturally! The natural isomorphism is most easily described by its inverse $\Psi_{X,G}$. Given $\psi \in \mathrm{Mor}_{\mathbf{Set}}(X, UG)$, i.e., a set mapping $\psi : X \to G$, we know that there is a unique group homomorphism $\phi : FX \to G$ such that $\phi \circ i_X = \psi$. We define $\Psi_{X,G}(\psi) = \phi$. To prove the naturality, let $f : X' \to X$ and $g : G \to G'$ be morphisms; we must check that the following two diagrams commute:

$$\begin{array}{rccc} \mathrm{Mor}(Ff, 1): & \mathrm{Mor}(FX, G) & \to & \mathrm{Mor}(FX', G) \\ & \Psi_{X,G} \uparrow & & \Psi_{X',G} \uparrow \\ \mathrm{Mor}(f, U1): & \mathrm{Mor}(X, UG) & \to & \mathrm{Mor}(X', UG) \end{array}$$

$$\begin{array}{rccc} \mathrm{Mor}(F1, g): & \mathrm{Mor}(FX', G) & \to & \mathrm{Mor}(FX', G') \\ & \Psi_{X',G} \uparrow & & \Psi_{X',G'} \uparrow \\ \mathrm{Mor}(1, Ug): & \mathrm{Mor}(X', UG) & \to & \mathrm{Mor}(X', UG') \end{array}$$

The first one commutes because for any $\psi \in \mathrm{Mor}(X, UG)$, the following diagram commutes:

$$\begin{array}{rccc} i_{X'}: & X' & \longrightarrow & FX' \\ & f \downarrow & & Ff \downarrow \\ i_X: & X & \longrightarrow & FX \\ & \psi & \searrow & \phi \downarrow \\ & & & G \end{array}$$

commutes, where $\phi = \Psi_{X,G}(\psi)$, and so $\phi \circ Ff = \Psi_{X',G}(\psi \circ f)$. The second commutes because for any $\psi \in \mathrm{Mor}(X', UG)$, the diagram

$$\begin{tikzcd} i_{X'} : X' \arrow[r] \arrow[dr, "\psi"'] & FX' \arrow[d, "\phi"] \\ & G \arrow[d, "g"] \\ & G' \end{tikzcd}$$

commutes, where $\phi = \Psi_{X',G}(\psi)$, and so $g \circ \Psi_{X',G}(\psi) = \Psi_{X',G'}(Ug \circ \psi)$.

2. On **Set**, if B is a fixed set, then the functor $\cdot \times B$ is left adjoint to the functor $\mathrm{Mor}(B, \cdot)$. That is,

$$\operatorname*{Mor}_{\mathbf{Set}}(A \times B, C) \cong \operatorname*{Mor}_{\mathbf{Set}}(A, \operatorname*{Mor}_{\mathbf{Set}}(B, C)),$$

naturally in A and C. A natural isomorphism takes $f : A \times B \to C$ to the function $\Theta(f)$ defined by $(\Theta(f)(a))(b) = f(a, b)$ $(\in C)$. The verification is left to the reader.

(Note: an "additive" version of 2 is that if B is a fixed object in **Ab**, or more generally $_{\mathbf{R}}\mathbf{Mod}$, then $\cdot \otimes_R B$ is left adjoint to $\mathrm{Hom}_R(B, \cdot)$, these functors going from $\mathbf{Mod}_{\mathbf{R}}$ to **Ab** and back. When we get to tensor products, this fact will be important.)

If we dualize the notion of adjoint functor, we find that the dual of a left adjoint is a right adjoint and vice-versa, since dualizing simply means interchanging the arguments in each Mor.

Theorem. *Left adjoints preserve coproducts (more generally, colimits); and right adjoints preserve products (more generally, limits).*

Proof. We prove that right adjoints preserve products. That left adjoints preserve coproducts then follows by duality. The proofs for more general limits and colimits is similar but a bit more complicated.

First we need the following characterization of products:

Lemma. *Let $\{A_i\}_{i \in I}$ be a family of objects in the category $\mathbf{C}$. Let A be an object in $\mathbf{C}$ and $\pi_i : A \to A_i$ a morphism for each $i \in I$. Then A and the π_i are a product of the family $\{A_i\}_{i \in I}$ if and only if for each object B in $\mathbf{C}$, the following mapping is a bijection (in the category of sets):*

$$\prod_{i \in I} \mathrm{Mor}(B, \pi_i) : \operatorname*{Mor}_{\mathbf{C}}(B, A) \to \prod_{i \in I} \operatorname*{Mor}_{\mathbf{C}}(B, A_i).$$

The reader should unravel this assertion enough to see that it is precisely the definition of product! That is, every family of morphisms $B \to A_i$ gives a unique $B \to A$, etc., etc.

Now to prove the theorem, suppose that A and π_i give a product of the objects A_i, $i \in I$, in the category $\mathcal{D}$. Let $F : \mathbf{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathbf{C}$ be an adjoint pair, with G a right adjoint to F. We must show that GA and $G\pi_i$ give a product in $\mathbf{C}$. Thus we must show that for every object B in $\mathbf{C}$,

$$\prod_{i\in I} \mathrm{Mor}(B, G\pi_i) : \underset{\mathbf{C}}{\mathrm{Mor}}(B, GA) \to \prod_{i\in I} \underset{\mathbf{C}}{\mathrm{Mor}}(B, GA_i)$$

is a bijection in **Set**. But we know that

$$\prod_{i\in I} \mathrm{Mor}(FB, \pi_i) : \underset{\mathcal{D}}{\mathrm{Mor}}(FB, A) \to \prod_{i\in I} \underset{\mathcal{D}}{\mathrm{Mor}}(FB, A_i)$$

is a bijection. The natural isomorphism between $\mathrm{Mor}_{\mathbf{C}}(\cdot, G\cdot)$ and $\mathrm{Mor}_{\mathcal{D}}(F\cdot, \cdot)$ allows us to make a commutative square

$$\begin{array}{ccc} \mathrm{Mor}_{\mathbf{C}}(B, GA) & \longrightarrow & \prod_{i\in I} \mathrm{Mor}_{\mathbf{C}}(B, GA_i) \\ \Theta_{B,A} \uparrow & & \prod_{i\in I} \Theta_{B,A_i} \uparrow \\ \mathrm{Mor}_{\mathcal{D}}(FB, A) & \longrightarrow & \prod_{i\in I} \mathrm{Mor}_{\mathcal{D}}(FB, A_i) \end{array}$$

and thus deduce the bijectivity of one of the displayed mappings from the other.

As a sample application: the free functor $F : \mathbf{Set} \to \mathbf{Gp}$ carries disjoint unions to free products (these being the coproducts in these two categories). The free functor $F : \mathbf{Set} \to \mathbf{Ab}$ or to ${}_{\mathbf{R}}\mathbf{Mod}$ carries disjoint unions to direct sums. And the forgetful functors from $\mathbf{Gp}$, $\mathbf{Ab}$, ${}_{\mathbf{R}}\mathbf{Mod}$ to $\mathbf{Set}$ preserve products. This is an "explanation" why products in these categories are based on the set-theoretic product.

Exercise. *Left adjoints preserve epics, and right adjoints preserve monics.*