Math 551, Assignment 8

Due December 12, in class

1. The proof given in lecture of the finite-dimensional Spectral Theorem can be adapted to the case of a self-adjoint transformation on a real vector space V. With one exception, the changes which have to be made are all quite trivial. This exception requires an extra line or two. Where is the difficulty and how do you get around it?

2. Below are several true statements about linear transformations of finite-dimensional vector spaces. Determine which are true for conjugate linear transformations of complex vector spaces, and modify the others to make them true for such transformations.

a) The composite of linear transformations, when defined, is a linear transformation.

b) If $T : V \to W$ is a linear transformation, then $\ker T$ and $\operatorname{im} T$ are subspaces, and $V/\ker T \cong W$.

c) If $T : V \to W$ is a linear transformation, then $T^*(f) = f \circ T$ defines a linear transformation $T^* : W^* \to V^*$.

3. Let A be a real symmetric matrix. Let V be a real vector space with bilinear form $\mathcal{B}$, and suppose that A is the matrix of $\mathcal{B}$ with respect to some basis $\{e_1, \ldots, e_n\}$ of V. (That is, $a_{ij} = \mathcal{B}(e_i, e_j)$.) Show that the following conditions are equivalent:

a) The eigenvalues of A are positive real numbers.

b) $\mathcal{B}$ is positive definite.

c) The principal minors of A are positive real numbers. (A principal minor of A is the determinant of a square submatrix A' of A such that the main diagonal of A' is part of the main diagonal of A.) (Hint. In showing c) implies a), express the coefficients of χ_A in terms of principal minors. In showing that a) and b) imply c), keep in mind that the restriction of $\mathcal{B}$ to any subspace is still positive definite.)

Such a matrix is called **positive** or positive definite.

4. Let R be a commutative ring and M an R-module. Construct an isomorphism of R-modules: $\operatorname{Hom}_R({}_RR, M) \cong M$, and show that it is natural. What has to be changed if R is not assumed to be commutative?

5. Show that in the categories **Ab** and **Gp**, "monic" is equivalent to "injective" and "epic" is equivalent to "surjective". Show however that in the category of rings, the inclusion $\mathbf{Z} \to \mathbf{Q}$ is epic. (Hint for epics in **Gp**. Given G and a proper subgroup H, look for morphisms $G \to \Sigma_{\{*\}\cup(G/H)}$, where $*$ is an additional point.)

6. Construct a left adjoint to the forgetful functor $U : \mathbf{Ab} \to \mathbf{Gp}$.

7. Let $\mathbf{C}$ be the category of "pointed rings", i.e., the objects are the pairs (R, x), R a ring and $x \in R$; and an arrow $(R, x) \to (S, y)$ is a ring homomorphism $\phi : R \to S$ such that $\phi(x) = y$. Find an initial object in $\mathbf{C}$.

8. Show that left adjoints preserve epics and right adjoints preserve monics.