Math 551, Assignment 6

Due Thursday, November 9, in class

1. Fix a prime p. Let $G = \prod_{n=1}^{\infty} \mathbf{Z}_{p^n}$, and let $H = \oplus_{n=1}^{\infty} \mathbf{Z}_{p^n} \leq G$. Show that for every $g \in T(G)$ there exists $g_1 \in T(G)$ such that $g - pg_1 \in H$. Conclude that $T(G)/H$ is not the direct sum of cyclic groups.

2. Let G be a finitely generated free abelian group, and let $H \leq G$. Show that there is a unique subgroup $H^* \leq G$ such that (a) $H \leq H^*$; (b) $\mathrm{rank}(H) = \mathrm{rank}(H^*)$; and (c) $G = H^* \oplus K$ for some subgroup K.

3. An elementary matrix is a square matrix which is the same as the identity matrix except possibly in a single off-diagonal entry. Let R be a Euclidean domain. Show that a square matrix $A \in R^{m\times m}$ satisfies $\det(A) = 1$ if and only if A is the product of elementary matrices in $R^{m\times m}$. Deduce that A is invertible in $R^{m\times m}$ if and only if A is the product of elementary matrices and a diagonal matrix whose entries are units in A.

4. For any $m \times n$ matrix A (square or not), and any positive integer $r \leq \min(m, n)$, an $r \times r$ minor of A is defined as an $r \times r$ matrix obtained by deleting some rows and columns of A. If A has all its entries in a Euclidean domain R, define $\Delta_r(A)$ to be the g.c.d. of the determinants of all its $r \times r$ minors (this is well-defined up to association). Use the previous exercise and the fundamental theorem to show that the invariant factors of A are

$$\Delta_1(A), \Delta_2(A)/\Delta_1(A), \cdots, \Delta_r(A)/\Delta_{r-1}(A),$$

where r is the greatest integer such that $\Delta_r(A) \neq 0$, i.e., r is the rank of the matrix A. [A consequence is that for two $m \times n$ matrices over the Euclidean domain R, $A \sim B \iff \Delta_r(A)$ and $\Delta_r(B)$ are associates for each $1 \leq r \leq \min(m, n)$.]

5. Let $H \leq G$, with G and H both free abelian groups of the same rank. Let A be the matrix of the inclusion mapping $\alpha : H \to G$ with respect to some bases B and B' of G and H respectively (i.e. the columns of A are the B-coefficients of the elements of H). Show that $|G/H| = \pm \det(A)$.

6. Isomorphism-invariant data obtained from all objects in a class (e.g. all abelian groups) are called **invariants** of those objects. A set of invariants is called **complete** when it is true that two objects are isomorphic if and only if they have the same invariants. Which of the following sets of invariants of a finite abelian group G is complete? Give proof or counterexample.

a) $\{|G|$, the exponent of G, the rank of $G\}$ (rank=smallest number of direct summands possible in an expression for G as the direct sum of cyclic groups).

b) $\{n_i \mid i \in \mathbf{Z}^+\}$, where for any positive integer i, n_i is the number of elements of G of order i.

7. Let M be a finitely generated module over a PID R, and let N be a submodule of M. Relate the invariant factors of N to those of M.

8. Show that the following hold for arbitrary finitely generated modules M, N, P over a PID: (a) $M \oplus P \cong N \oplus P$ implies $M \cong N$; (b) If M is isomorphic to a submodule of N, and N is isomorphic to a submodule of M, then $M \cong N$. Give counterexamples to show that the assumption of finite generation is necessary.

9. Let M be an abelian group. Show that the abelian group $\mathrm{Hom}(M, M)$, together with the operation of composition of functions, is a ring (with 1). Show that an R-module is an abelian group M together with a ring homomorphism $\phi : R \to \mathrm{Hom}(M, M)$, i.e., an R-module determines such a pair (M, ϕ), and such a pair determines an R-module. Then give a 1-line proof that if M is an R-module and A is an ideal such that $am = 0$ for all $a \in A$ and $m \in M$, then M is an R/A-module.

10. Let G be a finite subgroup of $GL_n(\mathbf{Q})$, where $\mathbf{Q}$ is the field of rational numbers. Show that G is conjugate (in $GL_n(\mathbf{Q})$) to a subgroup of $GL_n(\mathbf{Z})$. (Hint. Take any $\mathbf{Q}$-basis B of the $\mathbf{Q}$-vector space $\mathbf{Q}^n$ of $n \times 1$ column vectors, and consider the abelian group generated by all gb, $g \in G$, $b \in B$.)