Math 551, Assignment 4

Due Thursday, Oct. 12 in class

1. Let $K \triangleleft H \leq G$ and $N \triangleleft G$. Show that $KN \triangleleft HN$ and that HN/KN is a quotient of H/K (that is, there exists a surjective homomorphism $H/K \to HN/KN$).

2. Let $\mathbf{N}$ be the set of natural numbers. Let $\Sigma^o_{\mathbf{N}}$ be the set of all permutations of $\mathbf{N}$ with finite support. Define $A^o_{\mathbf{N}}$ and show that $A^o_{\mathbf{N}}$ is simple. (Hint: represent $A^o_{\mathbf{N}}$ as a union of finite alternating groups.)

3. Let $N \triangleleft G$. Show that G has a composition series if and only if both N and G/N have composition series.

On the other hand, give an example of a group G and a subgroup $H \leq G$ such that G has a composition series but H does not.

4. If $K \triangleleft G$ and $L \triangleleft G$, show that $G/(K \cap L)$ is isomorphic to a subgroup of $G/K \times G/L$.

5. Suppose $H_1, \ldots, H_n$ are subgroups of G and $|G : H_i|$ is finite for each $i = 1, \ldots, n$. Show that the index

$$\left| G : \bigcap_{i=1}^{n} H_i \right|$$

is finite and give a bound for it which is a function of the indices $|G : H_1|, \ldots, |G : H_n|$.

6. Let G be any simple group such that $|G| = 60$. Let $H = A_6$.

a) Construct an injective homomorphism $\phi : G \to H$ such that for any element $x \in G$ of order 3, $\phi(x)$ is the product of two disjoint 3-cycles.

b) Then construct an injective homomorphism $\psi : H \to A_{H/\phi(G)}$ such that every element of $\psi(\phi(G))$ fixes at least one point (of $H/\phi(G)$).

c) Conclude that any simple group of order 60 is isomorphic to A_5.

d) Conclude also that A_6 has an automorphism α such that $\alpha((123)) = (123)(456)$. (Thus α is not induced by conjugation by any element of Σ_6. !)

7. Show that A_5 contains elements x, y and z of orders 2, 3 and 5 respectively such that $xyz = 1$. On the other hand, show that no solvable group can contain such a triple of elements. Generalize the second statement from $(2, 3, 5)$ to other triples of integers.

8. Given two groups K and H and a homomorphism $\phi : H \to \mathrm{Aut}(K)$, one can define a group $G = H \ltimes_\phi K$ (the semidirect product of K by H via ϕ) as follows: the underlying set of G is the Cartesian product $K \times H$, and the multiplication is

$$(k_1, h_1)(k, h) = (k_1\phi(h_1)(k), h_1 h).$$

(The idea is that G should consist of "products" kh, $k \in K$, $h \in H$, with the rule $hkh^{-1} = \phi(h)(k)$ for moving h's past k's.)

Verify that G is a group, and there are two subgroups of G which can be identified with K and H in such a way $G = KH$, $K \cap H = 1$ and $K \triangleleft G$; moreover ${}^h k = \phi(h)(k)$ for all $h \in H$ and $k \in K$. Usually the notation $H \ltimes K$ is used, even though it omits essential information.

The holomorph of a group G is defined to be the semidirect product $\mathrm{Aut}(G) \ltimes G$, with ϕ being the the identity mapping. What is the holomorph of $\mathbf{Z}_2 \times \mathbf{Z}_2$?