Math 551, Assignment 2

Due Thursday, Sept. 28 in class

1. An action of a group G on a set Ω is said to be *doubly transitive*, or 2-*transitive*, if and only if for every (α, β) and $(\gamma, \delta) \in \Omega \times \Omega$ such that $\alpha \neq \beta$ and $\gamma \neq \delta$, there exists $g \in G$ such that $g\alpha = \gamma$ and $g\beta = \delta$.

a) Show that if $|\Omega| > 2$, then G is doubly transitive on Ω if and only if G is transitive on Ω and also G_α is transitive on $\Omega - \{\alpha\}$ for some $\alpha \in \Omega$.

b) Let n be an integer such that $n > 1$. Show that the natural action of $SL_n(\mathbf{C})$ on $\mathbf{C}^n - \{0\}$ (i.e. the set of all nonzero $n \times 1$ column vectors) is transitive but not doubly transitive, while the natural action of $SL_n(\mathbf{C})$ on the set of 1-dimensional subspaces of $\mathbf{C}^n$ is doubly-transitive.

2. The quaternion group Q_8 can be defined as the subgroup $\left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right\rangle$ of $SL_2(\mathbf{C})$. Demonstrate the following: a) $|Q_8| = 8$; b) Q_8 contains a unique element of order 2; c) Every subgroup of Q_8 other than Q_8 itself is cyclic; d) $Q_8 \not\cong D_8$.

3. Let G be a group of order 264. What can you say about the number of elements of order 11 in G? Can these elements all be conjugate in G?

4. Show that if p is a prime divisor of the finite group G, then the number of subgroups of G of order p is congruent to $1 \bmod p$. (Hint. Begin by adapting the proof of the corresponding part of Sylow's Theorem.)

5. Let $G = \Sigma_4$. Write down representatives of the conjugacy classes of G (i.e. exactly one element from each conjugacy class). For each representative g which you have written, compute $|C_G(g)|$ and give generators of $C_G(g)$. Do the same for A_4. (The counting principle is useful here!)

6. a) Let H be any subgroup of Σ_n. Show that $|H : H \cap A_n| = 1$ or 2.

b) Let $g \in A_n$. Show that each Σ_n-conjugate of g (i.e. each permutation of the same cycle shape as g) is already conjugate to g <u>by an element of A_n</u> if and only if g centralizes some odd permutation in Σ_n.

c) Give an example of an element $g \in A_{15}$ for which the conditions in b) are false.

7. Suppose that $g, h \in \Sigma_n$ and $gh = hg$.

a) Show that if Ω_1 is an orbit of $\langle g \rangle$ on $\{1, 2, \ldots, n\}$, then $h\Omega_1$ is also such an orbit.

b) Express the orbits of $C_{\Sigma_n}(g)$ in terms of the cycle decomposition of g.