Math 551, Assignment 1

Due Thursday, Sept. 14 in class

1. If X and Y are two sets, finite or infinite, we say that $|X| = |Y|$ if and only if there exists a bijection between X and Y. Show that if $|X| = |Y|$, then $\Sigma_X \cong \Sigma_Y$.

2. Let the finite group G act on the finite set Ω. For each $g \in G$ let $\theta(g) = |Fix_\Omega(g)|$. Show that $\sum_{g \in G} \theta(g) = k|G|$, where k is the number of orbits of G on Ω. (Hint. In the transitive case, count the number of pairs (g, α) such that $g\alpha = \alpha$, and do the count in two different ways.)

3. Suppose that G acts on Ω. For any integer n, let $\Omega^n = \Omega \times \cdots \times \Omega$ (n factors). Define in a natural way an action of G on Ω^n. If $G = \Sigma_\Omega$, then how many orbits does G have on Ω^n for $n = 2, 3, 4$? You may assume that Ω is sufficiently large, but do say how large that is.

4. Let G and H be cyclic groups, neither of which is the trivial group (of order 1). Give necessary and sufficient conditions for the direct product $G \times H$ to be cyclic.

5. Let $G = \mathbf{Z}_n$. Let $R = \mathbf{Z}/n\mathbf{Z}$ be the *ring* of integers modulo n, and $R^\times$ the group of units (multiplicatively invertible elements) of R, under the operation of multiplication. Show that $\mathrm{Aut}(G) \cong R^\times$.

6. Show that a group of exponent 2 is necessarily abelian, but construct a non-abelian subgroup of $SL_3(\mathbf{Z}/3\mathbf{Z})$ which has order 3^3 and exponent 3.

7. Show that $SL_2(\mathbf{Z}) = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right\rangle$. Conclude that $PSL_2(\mathbf{Z})$ is generated by an element of order 2 and an element of order 3. ($PSL_2(\mathbf{Z})$ is the group of transformations of $\mathbf{C} \cup \{\infty\}$ of the form $z \mapsto (az + b)/(cz + d)$ with $a, b, c, d \in \mathbf{Z}$ and $ad - bc = 1$.)

8. Show that if a group G is generated by two elements g, h of order 2, then G has a cyclic subgroup H such that $|G : H| = 2$ and $g, h \notin H$. What are the possible isomorphism types for G? Show that $PSL_2(\mathbf{Z})$ is not generated by two elements of order 2.