These problems cannot cover everything from the course; I suggest that you also review the midterm exams, their review sheets, and homework problems, particularly those on assignments 12 and 13, which primarily cover material that has not yet appeared on an exam..

Note: problems 1-8 below are the problems which made up the 2006 final in this course.

1. Let $y(t)$, $t \geq 0$, be the solution of

$$y''(t) - 6y'(t) + 9y(t) = f(t), \quad y(0) = 0, \quad y'(0) = 1.$$

Let $Y(s)$ be the Laplace transform of y and $F(s)$ be the Laplace transform of f.

(a) Find an expression for $Y(s)$ in terms of $F(s)$.

(b) Find $y(t)$ if $f(t) = \delta(t-1)$.

2. Consider the Sturm-Liouville problem

$$y'' + \lambda y = 0, \qquad y(0) + 2y'(0) = 0, \quad y(1) = 0.$$

(a) Is $\lambda = 0$ an eigenvalue.? If yes, give an associated eigenfunction. If not, show why not.

(b) Find a condition on $\lambda > 0$ that it must satisfy to be an eigenvalue, and for λ satisfying this condition write down an associated eigenfunction.

(c) There are an infinite number of positive eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$. Show graphically that $\pi^2 < \lambda_1 < (3\pi/2)^2$.

3. (a) Find the general solution to

$$\begin{aligned} 4u_{xx}(x,t) = u_t(x,t), \qquad 0 < x < \pi, \qquad t > 0; \\ u_x(0,t) = 0, \quad u(\pi,t) = 0, \qquad t > 0. \end{aligned}$$

(b) Find the solution $u(x,t)$ to

$$\begin{aligned} 4u_{xx}(x,t) = u_t(x,t) + 1, \qquad 0 < x < \pi, \qquad t > 0; \\ u_x(0,t) = 1, \quad u(\pi,t) = 1, \qquad t > 0; \\ u(x,0) = 0. \end{aligned}$$

You must give an explicit integral formula for the coefficients in your solution but you do not need to evaluate the integrals explicitly.

4. Consider the partial differential equation with boundary condition

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \qquad < x < \infty, \quad y > 0; \tag{1}$$

$$u(x,0) = f(x), \qquad -\infty < x < \infty. \tag{2}$$

We want to solve this equation in the upper half-plane, $y \geq 0$. Let

$$\hat{u}(\omega, y) = \int_{-\infty}^{\infty} e^{-i\omega x} u(x,y)\, dx$$

denote the Fourier transform of u in the x variable, for each $y \geq 0$.

(a) Fourier transform (1) to find an equation that $\hat{u}(\omega, y)$ must satisfy if u solves (1).

(b) Show that an expression of the form $\hat{u}(\omega, y) = A(\omega)e^{-|\omega| y}$, where is A is a function of ω only, satisfies the equation derived in (a).

(c) From (b) we know that the Fourier transform of a solution u to (1) has the form $\hat{u}(\omega, y) = A(\omega)e^{-|\omega| y}$. Identify what $A(\omega)$ must be if u also satisfies the condition (2). Calculate the Fourier inverse of $\hat{u}(\omega, y) = A(\omega)e^{-|\omega| y}$ with the A you found in (b), to find a solution $u(x,y)$ of (1) and (2) in the form of an integral involving f.

5. (a) Find $f(x)$ if the Fourier transform of f is $\hat{f}(\omega) = e^{-(\omega+1)^2}$.

(b) Find the Fourier transform of $e^{-|x-3|} + e^{-2|x|}$.

6. Consider the following differential equation for $y(x)$, $x > 0$:

$$2x^2y'' + xy' - (1 + x^2)y = 0. \tag{3}$$

(a) Explain why $x = 0$ is a regular singular point of this equation.

(b) We want to solve (3) by the Frobenius method. Find the indicial equation and determine its roots r_1 and r_2.

(c) Let $y_1(x)$ be the Frobenius method solution corresponding to the larger root and with $a_0 = 1$. Find y_1 explicitly out to its first three non-zero terms.

(d) Write down the general recursion relation (relating a_n to previous a_k) for determining the coefficients of the Frobenius method solutions.

7. Consider the wave equation

$$u_{xx} = u_{tt} - u, \qquad 0 < x < 1, \quad t > 0; \tag{4}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \qquad t > 0 \tag{5}$$

$$u(x,0) = 0, \quad u_t(x,0) = x, \qquad 0 < x < 1. \tag{6}$$

(a) Write down equations that must be satisfied by $X(x)$ and $T(t)$ if $u(x,t) = X(x)T(t)$ is to be a product solution to (4).

(b) Find all possible separable solutions to (4) and (5), and write down the general solution to these two equations.

(c) Explain how to choose the coefficients of the general solution to solve (4) and (6) together with the initial conditions given in (6). You should give explicit formulas, but these may have integrals in them.

8. This problem is about the nonlinear system in the plane:

$$x' = y, \qquad y' = -x + x^3 + y.$$

Its singular points are $x = -1$, $y = 0$, and $x = 1$, $y = 0$, and $x = 0$, $y = 0$.

(a) Classify the type and stability of the singular points $(x_0, y_0) = (1, 0)$ and $(x_1, y_1) = (0, 0)$, in so far as possible using linearization. If the linearization method does not fully classify the singular point, indicate what possibilities it does allow.

(b) Sketch the phase portrait for the linearized system at the singular point $(x_0, y_0) = (1, 0)$. Include arrows of flow directions and eigenvectors, where appropriate. You may leave out isoclines. Do the same for the singular point $(x_1, y_1) = (0, 0)$.

9. Solve Laplace's equation in the rectangle $0 \le x \le 4$,, $0 \le y \le 3$, with the boundary conditions:

(a) $u(0, y) = 0$, $u(4, y) = 1$, $u(x, 0) = 0$, $u(x, 3) = 0$;

(b) $u(0, y) = 0$, $u(4, y) = 1$, $u(x, 0) = 0$, $u(x, 3) = 2\sin 3\pi x$;

(c) $u_x(0, y) = 0$, $u(4, y) = 0$, $u(x, 0) = \cos 7\pi x/8$, $u(x, 3) = 0$.