A formula sheet will be provided with the exam; it is posted on the web page.

1. (a) Find the general solution of $\mathbf{z}' = A\mathbf{z}$, where $\mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} -4 & 4 \\ 1 & -4 \end{pmatrix}$.

(b) Give a careful drawing of the phase plane ($xy$-plane) for this system, showing enough trajectories to indicate qualitatively the motion in each region of the plane, as well as any "special" (straight-line) trajectories. Your trajectories should be marked with arrowheads giving the direction in which the solution moves as $t$ increases.

2. Consider the system

$$x' = y - 1, \qquad y' = y - x^2.$$

(a) Determine its singular (equilibrium) points find the linearized system near each. For each singular point, determine the type of its linearized behavior (focus, node, etc.) and whether it is unstable, stable, or asymptotically stable for the linearized problem.

(b) Answer the same questions—type, stability—about the nature of these singular points in the original, nonlinear system, if you have the information to do so, or explain why you cannot.

(c) Sketch the phase plane for this system, showing nullclines (curves on which the solution curves have horizontal or vertical tangents), direction of motion as the solutions cross the nullclines, and direction of motion in the regions separated by the nullclines.

(d) Combine the information from (a) and (c) to give a sketch of the trajectories in the phase plane, showing all significant features.

3. Determine the equation of the phase trajectories of the system

$$x' = xy, \qquad y' = x^2$$

and sketch the phase plane with enough trajectories to show the essential features. Use arrows to indicate the direction of motion along the trajectories.

4. In this problem we consider functions defined for $0 \le x \le \pi$ with the inner product

$$\langle f, g \rangle = \int_0^\pi f(x)g(x)\,dx.$$

(a) Show that the constant function 1 and the function $\cos x$ are orthogonal.

(b) Let

$$f(x) = \begin{cases} 2, & \text{if } 0 \le x \le \pi/2 \\ 0, & \text{if } \pi/2 < x \le \pi. \end{cases}$$

Find constants $c$ and $d$ such that $g(x) = c + d\cos x$ is the linear combination of 1 and $\cos x$ which is best possible approximation to $f(x)$, in the sense that $\|g - f\|$ is as small as possible.

5. The function $f(x)$ defined for $0 \le x \le 1$ by

$$f(x) = \begin{cases} x, & \text{if } 0 \le x \le 1/3, \\ 1, & \text{if } 1/3 < x \le 1 \end{cases}$$

has a quarter-range cosine series on this interval. (You do not have to give this series or compute its coefficients.) The series actually converges for all $x$ to a certain extension $g(x)$ of $f(x)$.

(a) Plot $g(x)$, showing several periods. Don't worry here about values of $g(x)$ at points where it is discontinuous.

(b) To what value does the Fourier series in (a) actually converge (i.e., what is its sum) when

(i) $x = 0$? (ii) $x = 1/3$? (iii) $x = 3$?

6. Consider the initial/boundary value problem for the function $u(x, t)$:

$$\begin{gathered} 4u_{xx} = u_t, \quad 0 < x < 3,\ t > 0; \\ u(0,t) = 0, \quad u_x(3,t) = 0, \quad t > 0; \\ u(x,0) = 5, \quad 0 \le x \le 3, \end{gathered}$$

(a) Solve PDE and boundary conditions by separation of variables, that is, determine all nonzero product solutions $u(x,t) = X(x)T(t)$. Give details.

(b) What sort of Fourier series (HRC, HRS, QRC, QRS, or other) is appropriate for this problem? Explain briefly.

(c) Find explicitly the expansion of $u(x, 0)$ in the series you chose in (a). Simplify the coefficients to the extent possible.

(d) Give the solution $u(x, t)$ of the initial/boundary value problem (as an infinite series).

7. Consider the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad 0 < x < 3; \qquad y(0) = 0, \quad y(3) + y'(3) = 0.$$

You may assume (it is true) that no eigenvalues of this problem are negative.

(a) Show that 0 is *not* an eigenvalue.

(b) Find an equation which $\lambda > 0$ must satisfy to be an eigenvalue.

(c) There are an infinite number of positive eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$. Give a graphical construction of these. Label your graph clearly: axes, curves, eigenvalues, etc. From your graph find a good approximation for $\lambda_n$ when $n$ is very large.

(d) Let $\phi_n(x)$ be the eigenfunction associated with $\lambda_n$. Give the formula for $\phi_n(x)$ (it will involve $\lambda_n$).

(e) A continuous function $f(x)$ defined for $0 \le x \le 3$ will have an expansion $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$. Give the formula for $c_n$ in terms of explicit integrals involving $\phi_n(x)$.