CHAPTER 3

Partial Differential Equations

In Chapter 2 we studied the homogeneous heat equation in both one and two dimensions, using separation of variables. Here we extend our study of partial differential equations in two directions: to the inclusion of inhomogeneous terms, and to the two other most common partial differential equations encountered in applications: the wave equation and Laplace's equation.

3.1 Inhomogeneous problems: the method of particular solutions

In this section we will study to inhomogeneous problems only for the one-dimensional heat equation on an interval, but the general principles we discuss apply to many other problems as well. The *homogeneous* version of the problem we consider is obtained by taking the general setup that we considered in Sections 2.2 and 2.3 but including more general boundary conditions of the kind we considered for Sturm-Liouville problems (see Section 7.7 of Greenberg):

$$\begin{array}{llr} \text{PDE:} & u_t(x,t) - \epsilon^2 u_{xx}(x,t) = 0, & 0 < x < L, \quad t > 0, \\ \text{BC:} & \begin{array}{l} \alpha\, u(0,t) + \beta\, u_x(0,t) = 0 \quad \text{and} \\ \gamma\, u(L,t) + \delta\, u_x(L,t) = 0, \end{array} & t > 0 \\ \text{IC:} & u(x,0) = f(x), & 0 < x < L, \end{array} \tag{3.1}$$

where α and β are not both zero, and neither are γ and δ. Note that we are now denoting the thermal diffusivity by ϵ^2 rather than as α^2, to avoid conflict with the use of α in specifying the boundary conditions. We have discussed in Sections 2.2 and 2.3 the standard way to solve (3.1): by *separation of variables.* There we considered only Dirichlet and Neumann boundary conditions, but the more general conditions in (3.1) are easily handled: they simply lead to Sturm-Liouville eigenvalue problems, which we have also discussed.

Remark 3.1: Before taking up the inhomogeneous version of (3.1) we discuss briefly the physical interpretation of the equations. Recall from Section 2.2 that the PDE describes the temperature $u(x,t)$, at (longitudinal) position x and time t, of a rod of length L, assumed to be of such small cross section that we can regard the temperature as depending only on x (i.e., not on coordinates transverse to the rod), and to have its lateral surface so well insulated that heat can flow into or out of the rod only through the ends. The boundary conditions control this heat flow through the ends. To understand how this comes about one uses the fact that the current or flux of heat energy in rod at position x and time t, $q(x,t)$, is proportional to the gradient of the temperature:

$$q(x,t) = -kAu_x(x,t). \tag{3.2}$$

Here k is a positive constant, called the *thermal conductivity* of the rod, and A is the cross sectional area of the rod; the negative sign indicates that heat flows in the direction opposite to the temperature gradient, that is, from hotter regions to cooler ones. Consider now the boundary condition at the left end of the rod: $\alpha\, u(0,t) + \beta u_x(0,t) = 0$.

- If $\beta = 0$ (and so $\alpha \neq 0$) then this is the *Dirichlet* boundary condition $u(0,t) = 0$: the left end of the rod is maintained at temperature zero. One usually thinks of achieving this situation by putting the end of the rod in perfect thermal contact with a *heat reservoir* at this temperature. There is no direct control over the heat flux through this end.
- If $\alpha = 0$ (and so $\beta \neq 0$) then this is the *Neumann* boundary condition $u_x(0,t) = 0$. From (3.2), then, the heat flux at the left end, $q(0,t)$, is also zero, that is, this end is *insulated*: no heat can enter or leave the rod at $x = 0$. The actual temperature at the end does not enter into the boundary condition.
- If neither α nor β is zero then then this is a *mixed* or *Robin* boundary condition. Using (3.2) it may be written as

$$q(0,t) = \eta u(x,t), \qquad \text{with} \quad \eta = \frac{kA\alpha}{\beta}, \tag{3.3}$$

  so that the mixed boundary condition describes a situation in which the heat flux through the end of the rod is proportional to the temperature there. If $\eta < 0$ then heat leaves the rod when $u(x,0) > 0$ and enters it when $u(x,0) < 0$. Then (3.3) is a physically reasonable condition which may be thought of as modeling to a thin layer of imperfect insulation separating the end of the rod from a heat reservoir at zero temperature: heat leaks through the insulation and the simplest assumption is that it does so at a rate proportional to the temperature difference between the end of the rod and the reservoir. This is consistent with *Newton's law of cooling*, although that is usually used to model convective cooling, or with *Fourier's law of heat conduction*, if the insulation is so thin that one may assume that its thermal state reacts instantaneously to changes in $u(0,t)$. If $\eta > 0$ then heat is entering the rod when $u(x,0) > 0$ and leaving when $u(x,0) < 0$; this is physically an artificial system but is useful for mathematical exposition.

Similar considerations apply at the right end of the rod, but with the mixed boundary condition $q(0,t) = \eta u(x,t)$ it is now $\eta > 0$ which is physically realistic.

The boundary value problem (3.1) is called homogeneous because the PDE and BC contain only terms proportional to u or its derivatives. The *inhomogeneous* version of the problem is more general:

$$\begin{array}{lccr} \text{PDE:} & u_t(x,t) - \epsilon^2 u_{xx}(x,t) = F(x,t), & 0 < x < L, \quad t > 0, & \\ \text{BC:} & \begin{array}{c} \alpha\, u(0,t) + \beta\, u_x(0,t) = g(t) \quad \text{and} \\ \gamma\, u(L,t) + \delta\, u_x(L,t) = h(t). \end{array} & t > 0 & (3.4) \\ \text{IC:} & u(x,0) = f(x), & 0 < x < L, & \end{array}$$

Here the (potentially) nonzero terms $F(x,t)$, $g(t)$, and $h(t)$ are referred to as the *inhomogeneities* in the problem. (In a sense the nonzero initial condition in (3.1) also represents an inhomogeneity, but one does not usually use this terminology because if the initial condition were zero then the solution would be trivial and uninteresting: $u(x,t) = 0$ for all x, t.) Our method for solving (3.4) is similar to the standard method for solving inhomogeneous ODE's. We first find a *particular solution* which solves the PDE and the BC (or in some cases, just one of these—see Remark 3.4 below), and use this to reduce the problem to a homogeneous one, or at least to reduce the degree of inhomogeneity in the original problem. The simplest case occurs when the inhomogeneities are independent of time, that is, when g and h are constant and F is constant or depends only on x; in this case a particular solution which is also time-independent can often be found.

Rather than giving an exhaustive description of exactly how the method works in all cases we will concentrate on describing some examples.

Example 3.1: Suppose that the PDE in (3.4) is homogeneous but that there are time-independent inhomogeneous Dirichlet boundary conditions:

$$\begin{array}{lll} \text{PDE:} & u_t(x,t) - \epsilon^2 u_{xx}(x,t) = 0, & 0 < x < L, \quad t > 0, \\ \text{BC:} & u(0,t) = u_1, \qquad u(L,t) = u_2, & t > 0 \\ \text{IC:} & u(x,0) = f(x), & 0 < x < L, \end{array} \tag{3.5}$$

with u_1 and u_2 constant. Physically, this means that the two ends of the rod are held at the constant temperatures u_1 and u_2. We look for a *time independent* solution $v(x)$ of the PDE and BC, ignoring for the moment the IC. Substituting $v(x)$ into the PDE gives

$$v_t - \epsilon^2 v_{xx} = 0, \qquad \text{i.e.,} \qquad v'' = 0, \qquad \text{so} \qquad v(x) = A + Bx, \quad A, B \text{ constant.}$$

Thus $v(x) = A + Bx$ satisfies the PDE. It will satisfy the boundary conditions if

$$v(0) = A = u_1 \qquad \text{and} \qquad v(L) = A + LB = u_2;$$

these equations are easily solved to yield

$$v(x) = u_1 + (u_2 - u_1)\frac{x}{L}. \tag{3.6}$$

Now we write our original unknown $u(x,t)$ as a sum of the particular solution plus another function $w(x,t)$:

$$u(x,t) = v(x) + w(x,t).$$

The function $w(x,t)$ is our new unknown; it will be the solution of a problem similar to (3.4). Since both $u(x,t)$ and $v(x)$ satisfy the PDE (3.5), $w(x,t) = u(x,t) - v(x)$ will also:

$$w_t - \epsilon^2 w_{xx} = (u - v)_t - \epsilon^2 (u - v)_{xx} = (u_t - \epsilon^2 u_{xx}) - (w_t - \epsilon^2 w_{xx}) = 0 - 0 = 0. \tag{3.7}$$

Since $u(x,t)$ and $v(x)$ satisfy the same inhomogeneous Dirichlet BC (3.5), $w(x,t)$ will satisfy the *homogeneous* Dirichlet BC

$$w(0,t) = u(0,t) - v(0) = u_1 - u_1 = 0, \qquad w(L,t) = u(L,t) - v(0) = u_2 - u_2 = 0. \tag{3.8}$$

Finally, $w(x,t)$ will satisfy the initial condition

$$w(x,0) = u(x,0) - v(x) = f(x) - v(x). \tag{3.9}$$

In summary, $w(x,t)$ solves a problem with **homogeneous PDE and BC** and a **new IC**:

$$\begin{array}{rcc} \text{PDE:} & w_t(x,t) - \epsilon^2 w_{xx}(x,t) = 0, & 0 < x < L, \quad t > 0, \\ \text{BC:} & w(0,t) = 0 \quad \text{and} \quad w(L,t) = 0 & t > 0 \\ \text{IC:} & u(x,0) = f(x) - v(x), & 0 < x < L. \end{array} \tag{3.10}$$

But we know how to solve (3.10) with a half range sine series (see Section 2.2):

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-(\epsilon n\pi/L)^2 t} \tag{3.11}$$

with

$$b_n = \frac{2}{L}\int_0^L \big(f(x) - v(x)\big)\sin\frac{n\pi x}{L}\,dx. \tag{3.12}$$

Putting together (3.6) and (3.11) have the solution to (3.5):

$$u(x,t) = v(x) + w(x,t) = u_1 + (u_2 - u_1)\frac{x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-(\epsilon n\pi/L)^2 t}. \tag{3.13}$$

Note that the coefficients b_n are calculated in (3.12) from the **new** initial condition (3.9).

Remark 3.2: The steady state. One important question for the solution of the PDE is the long-time behavior and in particular whether or not the temperature reaches some steady state as $t \to \infty$. For the solution $u(x,t)$ found in (3.13) above the answer is yes; the exponentials $e^{-(\epsilon n\pi/L)^2 t}$ all vanish as $t \to \infty$ and so

$$u(x,t) \longrightarrow v(x) = u_1 + (u_2 - u_1)\frac{x}{L} \quad \text{as} \quad t \to \infty.$$

This the steady state solution in this case is just the particular solution that we used during the solution process. The notions of steady state solution and particular solution are closely related; when the particular solution used is time independent it will in fact always be *a* steady state solution, but in general it may not be the one relevant for our problem—that is, the $t \to \infty$ limit of $u(x,t)$. See Remark 3.3 below.

Example 3.2: Now we consider the same problem as in Example 3.1 but with other and more general boundary conditions. We will discuss primarily the problem of finding the particular solution; once that is done, the completion of the solution goes as in Example 3.1.

Consider then the problem

$$\begin{array}{llr} \text{PDE:} & u_t(x,t) - \epsilon^2 u_{xx}(x,t) = 0, & 0 < x < L, \quad t > 0, \\ \text{BC:} & \begin{array}{c} \alpha\, u(0,t) + \beta\, u_x(0,t) = g \quad \text{and} \\ \gamma\, u(L,t) + \delta\, u_x(L,t) = h, \end{array} & t > 0 \\ \text{IC:} & u(x,0) = f(x), & 0 < x < L, \end{array} \tag{3.14}$$

with g and h constant. As in Example 3.1 we look for a time independent solution $v(x)$ of the PDE and BC, and again this will have the form $v(x) = A + Bx$, where A and B must be chosen to satisfy the BC. We considered in Example 3.1 the case of Dirichlet boundary conditions at each end of the rod, obtained when $\beta = \delta = 0$; we will abbreviate this as Dirichlet/Dirichlet BC. Other simple cases may be discussed similarly:

Dirichlet/Neumann BC ($\beta = \gamma = 0$): Let us write the BC in the form

$$u(0,t) = u_1, \qquad u_x(L,t) = Q_2. \tag{3.15}$$

(The use of the letter Q here is meant to suggest a heat flux, although in fact we know from (3.2) the the heat flow at the end of the rod is $q(L,t) = -kAQ_2$.) Applying the BC (3.15) to the solution $v(x) = A + Bx$ leads to $v(0) = A = u_1$, $v'(L) = B = Q_2$, and thus to the particular solution

$$v(x) = u_1 + Q_2 x.$$

Using this we can reduce (3.14) to a homogeneous problem with Dirichlet/Neumann BC, which we solve with a QRS series.

Neumann/Dirichlet BC ($\alpha = \delta = 0$): Now the BC are

$$u_x(0,t) = Q_1, \qquad u(L,t) = u_2. \tag{3.16}$$

Everything proceeds as before; the particular solution is

$$v(x) = u_2 + Q_1(L - x)x,$$

and the resulting a homogeneous problem is solved with a QRC series.

Neumann/Neumann BC ($\alpha = \gamma = 0$): This case is somewhat different from the preceding ones. The BC are

$$u_x(0,t) = Q_1, \qquad u_x(L,t) = Q_2. \tag{3.17}$$

The particular solution $v(X) = A + Bx$ must then satisfy

$$v'(0) = B = Q_1 \quad \text{and} \quad v'(L) = B = Q_2. \tag{3.18}$$

Two different cases must be considered.

Neumann/Neumann BC, Case 1: $Q_1 = Q_2 = Q$. In this case the solution of (3.18) is $B = Q$; A is undetermined and we may take it to be zero for simplicity, yielding the particular solution

$$v(x) = Qx.$$

The resulting homogeneous problem for $w(x,t) = u(x,t) - v(x)$ is solved with an HRC series, leading to

$$u(x,t) = v(x) + w(x,t) = Qx + a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} e^{-(\epsilon n\pi/L)^2 t}, \tag{3.19}$$

with

$$a_0 = \frac{2}{L}\int_0^L (f(x) - Qx)\,dx, \qquad a_n = \frac{1}{L}\int_0^L (f(x) - Qx)\cos\frac{n\pi x}{L}\,dx. \tag{3.20}$$

Remark 3.3: The steady state again. Note that from (3.19) we now have

$$u(x,t) \longrightarrow a_0 + Bx \quad \text{as} \quad t \to \infty;$$

the steady-state solution differs from our particular solution by the constant term a_0, found in (3.20). To understand what is going on physically here one may observe that since Q_1 and Q_2 are proportional to the energy fluxes at the two ends of the system, the condition $Q_1 = Q_2$ means that no net thermal energy enters or leaves the system at any time $t > 0$. Since the thermal energy at time t is proportional to $\int_0^L u(x,t)\,dx$, this integral must in fact be independent of time. This can be used to determine a_0 without solving the full problem; see Exercise 18.3.10(c) of Greenberg.

Neumann/Neumann BC, Case 2: $Q_1 \neq Q_2$. In this case the equations (3.18) for $v(x)$ **have no solution**; there is no time-independent steady state. Physically this is because we are imposing different heat fluxes at the two ends of the rod, so that the total thermal energy must either increase or decrease forever. This case is explored further in Exercise 18.3.10(c) of Greenberg, and a solution is sketched in Exercise 18.3.19.

Robin/Robin BC. We now discuss briefly the most general boundary conditions: Robin (i.e., mixed) BC at each end of the rod. That is, we study (3.14) with no assumptions about the coefficients α, β, γ, and δ. We still wish to use the the particular solution $v(x) = A + Bx$ of the PDE, for which the BC become

$$\begin{aligned} \alpha\, v(0) + \beta\, v'(0) &= \alpha A + \beta B = g,\\ \gamma\, v(L) + \delta\, v'(L) &= \gamma(A + BL) + \delta B = h, \end{aligned}$$

Rewriting these equations as

$$\begin{array}{rcrcl} \alpha\, A &+& \beta\, B &=& g,\\ \gamma\, A &+& (\gamma\, L + \delta)\, B &=& h, \end{array} \tag{3.21}$$

we see that A and B must satisfy a linear system of two equations in two unknowns. Such a system may or may not have a solution. If it does, we will have found our particular solution; if not, we will have to consider other methods. The system will have a solution for every g and h if and only if the determinant of the coefficients is nonzero, that is, if and only if

$$\alpha(\gamma L+\delta)-\beta\gamma\neq 0. \tag{3.22}$$

One checks easily that, as would be expected from the above discussions, (3.22) is satisfied for Dirichlet/Dirichlet, Dirichlet/Neumann, and Neumann/Dirichlet BC, but not for Neumann/Neumann BC. When (3.22) is not satisfied we will be able to find a time independent particular solution only for certain values of the right hand sides g and h, just as for Neumann/Neumann BC we could find a solution only for $Q_1 = Q_2$.

Example 3.3: We finally consider an example in which the PDE, as well as the BC, is homogeneous. For simplicity we consider homogeneous Dirichlet BC:

$$\begin{array}{llr} \text{PDE:} & u_t(x,t)-\epsilon^2 u_{xx}(x,t)=F, & 0<x<L, \quad t>0, \\ \text{BC:} & u(0,t)=0, \qquad u(L,t)=0, & t>0 \\ \text{IC:} & u(x,0)=f(x), & 0<x<L, \end{array} \tag{3.23}$$

with F constant (the same method would work if F depended on x but not on t). Since both the PDE and the BC are time independent we again look for a time-independent particular solution $v(x)$ satisfying

$$\begin{array}{llr} \text{ODE:} & v_t(x)-\epsilon^2 v_{xx}(x)=-\epsilon^2 v''(x)=F, & 0<x<L, \\ \text{BC:} & v(0)=0, \qquad v(L)=0, & (3.24) \end{array}$$

Solving the ODE $v''=-F/\epsilon^2$ gives $v(x)=-\dfrac{Fx^2}{2\epsilon^2}+A+Bx$, and imposing the boundary conditions implies that $A=0$ and $B=\dfrac{FL}{2\epsilon^2}$, so that

$$v(x)=\frac{FL}{2\epsilon^2}x-\frac{F}{2\epsilon^2}x^2$$

Now one writes $u(x,t)=v(x)+w(x,t)$ and proceeds as in Example 3.1, leading to

$$u(x,t)=v(x)+w(x,t)=\frac{FL}{2\epsilon^2}x-\frac{F}{2\epsilon^2}x^2+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}e^{-(\epsilon n\pi/L)^2 t}. \tag{3.25}$$

with

$$b_n=\frac{2}{L}\int_0^L\Big(f(x)-\frac{FL}{2\epsilon^2}x+\frac{F}{2\epsilon^2}x^2\Big)\sin\frac{n\pi x}{L}\,dx. \tag{3.26}$$

Remark 3.4: Sometimes it is convenient to find the particular solution in two or more steps. As a rather naive example, suppose that we had assumed that, because the given BC in Example 3.3 are inhomogeneous, we could ignore them in finding the particular solution —that is, find a solution only of the PDE. This would have led us to the particular solution $z(x) = -\frac{Fx^2}{2\epsilon^2}$, and now writing $u(x,t) = z(x) + w(x,t)$ (this is not the same $w(x,t)$ as above) we would find that w satisfied

$$\begin{aligned}
&\text{PDE:} && w_t(x,t) - \epsilon^2 w_{xx}(x,t) = 0, && 0 < x < L, \quad t > 0, \\
&\text{BC:} && w(0,t) = 0, \qquad w(L,t) = \frac{FL^2}{2\epsilon^2}, && t > 0 \\
&\text{IC:} && w(x,0) = f(x) - z(x), && 0 < x < L,
\end{aligned} \tag{3.27}$$

We have thus introduced inhomogeneous boundary conditions in the new problem. But (3.27) is just a special case of the problem treated in Example 3.1, and as in that example could be solved by introducing a second particular solution, that is, a solution of the PDE and BC in (3.27). The final result would have been the same. In this case our original procedure is obviously preferable, but sometimes it is not so clear how to find a single particular solution solving both the PDE and the BC, and the two-step process is natural. This is what is going on in Exercise 18.3.19 of Greenberg.