

Taylor series (with radii of convergence given):

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_0^{\infty} x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_0^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_0^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_0^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

The Gamma function. For $x > 0$, $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$.

If x is not 0 or a negative integer, $\Gamma(x+1) = x\Gamma(x)$.

If n is a non-negative integer, $\Gamma(n+1) = n!$. $\Gamma(1/2) = \sqrt{\pi}$.

The Method of Frobenius—solution forms:

$$y_1(x) : x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$

$$y_2(x) : x^{r_2} \sum_{n=0}^{\infty} b_n x^n \quad \text{or} \quad y_1(x)(\ln x) + x^{r_2} \sum_{n=1}^{\infty} c_n x^n \quad \text{or} \quad C y_1(x)(\ln x) + x^{r_2} \sum_{n=0}^{\infty} c_n x^n.$$

The Method of Frobenius—useful formula:

$$u'' + p(x)u' + q(x)u = \frac{C}{x^2} [y_1(x) - xp(x)y_1(x) - 2xy_1'(x)].$$

Bessel Functions.

A. The Bessel equation of order ν : $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$.

B. If u solves the Bessel equation of order ν , and $b > 0$, then

$$y(x) = x^{\nu/\alpha} u(\alpha\sqrt{bx}^{1/\alpha}) \quad \text{solves} \quad y'' + \frac{a}{x}y' + bx^{c-a}y = 0,$$

where

$$\alpha = \frac{2}{c-a+2} \quad \text{and} \quad \nu = \frac{1-a}{c-a+2}.$$

C. Bessel functions:

$$J_{\nu}(x) = \left(\frac{x}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k}$$

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k-\nu+1)} \left(\frac{x}{2}\right)^{2k}$$

$$Y_{\nu}(x) = \frac{(\cos \nu\pi)J_{\nu}(x) - J_{-\nu}(x)}{\sin \nu\pi}, \quad \text{if } \nu \neq 0, 1, 2, \dots$$

$$Y_n(x) = \lim_{\nu \rightarrow n} Y_{\nu}(x), \quad \text{if } n = 0, 1, 2, \dots$$