SUMMARY OF THE METHOD OF FROBENIUS

Consider the linear, homogeneous, second order equation:

$$y'' + p(x)y' + q(x)y = 0. \tag{1}$$

Suppose that $x = 0$ a *regular singular point*:

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad |x| < R_1, \qquad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n, \quad |x| < R_2, \qquad R_1, R_2 > 0.$$

Define $\gamma(r) = r(r-1) + p_0 r + q_0$; the *indicial equation* is

$$\gamma(r) = 0, \qquad \text{roots } r_1, r_2.$$

Case (i). r_1 and r_2 are distinct and do not differ by an integer. There are two linearly independent solutions:

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, \qquad y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n, \qquad a_0 = b_0 = 1. \tag{2}$$

Case (ii). $r_1 = r_2$. There is one solution $y_1(x)$ of the form given in (2), and a second solution with the form

$$y_2(x) = y_1(x)(\ln x) + x^{r_1} \sum_{n=1}^{\infty} c_n x^n. \tag{3}$$

Case (iii). $r_1 = r_2 + m$, m a positive integer. There is one solution $y_1(x)$ as in (2), and a second solution with the form

$$y_2(x) = C y_1(x)(\ln x) + x^{r_2} \sum_{n=0}^{\infty} d_n x^n, \qquad b_0 = 1. \tag{4}$$

The constant C may or may not be zero. One may assume that $d_m = 0$; see below.

FURTHER COMMENTS

1. Normalization. In these formulas we have "normalized" the solutions by choosing a_0, and also b_0 in Case (i) and D_0 in Case (iii), to have value 1. We could just as well have said only that they were nonzero, but it is convenient to have the solutions $y_1(x)$ and $y_2(x)$ completely defined.

2. Radius of convergence. All the power series in (2)–(4) are guaranteed to have radius of convergence at least as big as the smaller of R_1 and R_2.

3. Solution procedure, Case (i). The coefficients a_n of the solution $y_1(x)$ are determined by substituting the given expression (2) for $y_1(x)$ into (1) and then solving successive equations for a_1, a_2, ... These have the form (before we set $a_0 = 1$)

$$\gamma(n + r_1)a_n = \text{a linear combination of } a_0, a_1, \ldots, a_{n-1}. \tag{5}$$

The coefficients b_n of the second solution $y_2(x)$ in Case (i) are found similarly.

4. Solution procedure, Cases (ii) and (iii). In these cases one first finds $y_1(x)$. The solution $y_2(x)$ of (3) or (4) can be written as $y_2(x) = Cy_1(x)(\ln x) + u(x)$, where $C = 1$ in Case (ii) and C is to be determined in Case (iii), and in each case u is given by a series. Substituting this form into (1) one finds that $u(x)$ must satisfy the equation

$$u'' + p(x)u' + q(x)u = \frac{C}{x^2}\left[y_1(x) - xp(x)y_1(x) - 2xy_1'(x)\right]. \tag{6}$$

One then substitutes the form of the series for $u(x)$, as given in (3) or (4), into (6) and solves for $c_1, c_2, \ldots$ in Case (ii) or for C and $d_1, d_2, \ldots$ in Case (iii). The general structure of the equations will be similar to (5). In Case (ii) these will look like

$$\gamma(n + r_2)c_n = \text{a constant term plus a linear combination of } c_1, c_2, \ldots, c_{n-1}. \tag{7}$$

(Recall that $C = 1$ in Case (ii).) In Case (iii) we will have

$$\gamma(n + r_2)d_n = \text{a linear combination of } C \text{ and } d_1, d_2, \ldots, d_{n-1}. \tag{8}$$

In this case the constant C (which must be solved for) first appears on the right hand side of (7) when $m = n$; then $\gamma(m + r_2) = \gamma(r_1) = 0$ so that the left hand side vanishes (and d_m is not determined). Then C must be chosen to make the right hand side vanish also.

5. Additional free constants. Notice that there is no c_0 coefficient in (3). One could include a c_0 term in the solution, but the value of c_0 would not be determined by the equations; c_0 could be chosen freely. Choosing a nonzero value for the c_0, however, would amount to adding a multiple of $y_1(x)$ to the solution $y_2(x)$ as given in (3).

The situation for Case (iii) is similar. The coefficient d_m in (4) will not be determined during the solution process, and it is simplest to choose $d_m = 0$. Choosing a nonzero value for d_m again amounts to adding a multiple of $y_1(x)$ to the solution.

6. An ordinary point. An ordinary point of a differential equation may be considered, in some sense, as a special case of a regular singular point. If $x = 0$ is an ordinary point of (1) then the above analysis applies; one finds that $\gamma(r) = r(r - 1)$ and hence that $r_1 = 1$ and $r_0 = 0$: we are in Case (iii). However, we already know that in this case there are two linearly independent solutions, as power series in x, which do not contain $\ln x$; this means that necessarily $C = 0$.