1. Does $\displaystyle\int_0^\infty \frac{x^2+1}{x^4+4}\,dx$ converge or diverge? **Justify your answer.**

Solution: The integrand is well behaved everywhere so the only question is what happens as $x\to\infty$. When x is very large, $\dfrac{x^2+1}{x^4+4}\approx\dfrac{x^2}{x^4}=\dfrac{1}{x^2}$ so the convergence is the same as $\displaystyle\int_a^\infty \frac{dx}{x^2}$ (with $a>0$ arbitrary). But $\displaystyle\int_a^\infty \frac{dx}{x^2}=-\frac{1}{x}\bigg|_a^\infty=\frac{1}{a}$, so the integral converges.

2. Suppose that $f(x,y)=xy^2\sin(x^2y)$. Find f_x and f_{xy}.

Solution: From the product rule and the chain rule,

$$\begin{aligned}
f_x&=\frac{\partial f}{\partial x}=y^2\sin(x^2y)+(xy^2)(2xy)\cos(x^2y)=y^2\sin(x^2y)+2x^2y^3\cos(x^2y),\\
f_{xy}&=\frac{\partial}{\partial y}\frac{\partial f}{\partial x}=2y\sin(x^2y)+y^2(x^2\cos(x^2y))+6x^2y^2\cos(x^2y)+2x^2y^2(-x^2\sin(x^2y))\\
&=2y(1-x^4y)\sin(x^2y)+7x^2y^2\cos(x^2y).
\end{aligned}$$

3. Find the solution $y(x)$ of the initial value problem $y''+4y=0$, $y(0)=0$, $y'(0)=3$.

Solution: Since this is a constant coefficient linear equation we expect solutions of the form $y=e^{rx}$; substituting this form into the equation we find that r must satisfy $r^2+4=0$, so $r=\pm 2i$. The general solution is $y=c_1e^{2ix}+c_2e^{-2ix}$ or more simply $y=A\cos 2x+B\sin 2x$; imposing the initial conditions gives $y=(3/2)\sin 2x$.

4. Find the general solution $y(x)$ of the equation $yy'=(1+x)(1+y^2)$.

Solution: We separate variables (remember that $y'=dy/dx$) to obtain

$$\frac{y}{1+y^2}\,dy=(1+x)\,dx.$$

Integrating gives $\frac{1}{2}\log(1+y_2)=\frac{1}{2}(1+x)^2+C$ or $y=\pm\sqrt{e^{(x+1)^2+2C}-1}$.

5. Find the eigenvalues of the matrix $A=\begin{pmatrix}-4&1\\-3&0\end{pmatrix}$, and find an eigenvector for one of them.

Solution: The eigenvalues are the roots of the equation

$$\det(A-\lambda I)=\begin{vmatrix}-4-\lambda&1\\-3&-\lambda\end{vmatrix}=\lambda^2+4\lambda+3=0,$$

so $\lambda_1=-1$, $\lambda_2=-3$. To find an eigenvector $\mathbf{u}$ for λ_1 we solve $\begin{pmatrix}-4+1&1\\-3&1\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=0$, finding $\mathbf{u}=\begin{pmatrix}1\\3\end{pmatrix}$. Similarly, an eigenvector $\mathbf{v}$ for λ_2 is $\mathbf{v}=\begin{pmatrix}1\\1\end{pmatrix}$.

6. Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^\infty\frac{3^nx^{2n}}{n}$.

Solution: We try the ratio test: if $b_n=n3^nx^{2n}$ is a typical term of the series then

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)3^{n+1}x^{2(n+1)}}{n3^nx^{2n}}\right|=\lim_{n\to\infty}\left|\frac{(n+1)}{n}\right|3|x|^2=3|x|^2.$$

The series converges if this limit is less than 1, i.e., if $|x|<1/\sqrt{3}$; the radius of convergence is $1/\sqrt{3}$.