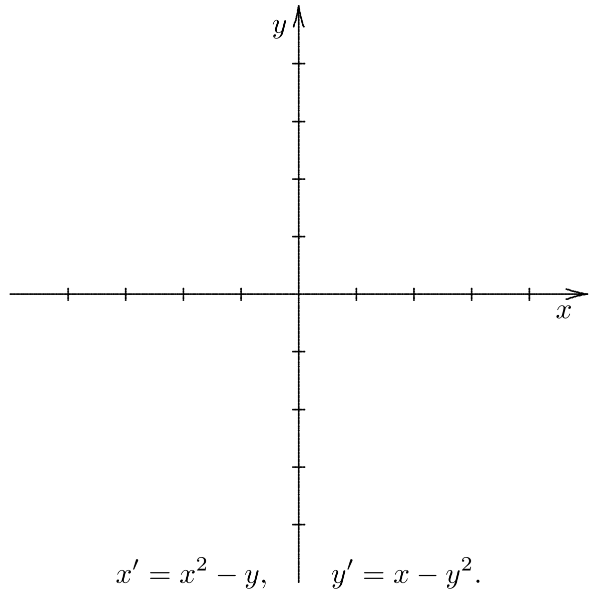
1. (a) Find the general solution of $\mathbf{z}' = A\mathbf{z}$, where $\mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 3 \\ -2 & -5 \end{pmatrix}$. **Be sure you actually give this solution** (which should involve two free constants).

(b) On the axes below give a careful drawing of the phase plane (xy-plane) for this system, showing enough trajectories to indicate qualitatively the motion in each region of the plane, as well as any "special" (straight-line) trajectories. Your trajectories should be marked with arrowheads giving the direction in which the solution moves as t increases.



2. Consider the system

$$x' = x^2 - y, \qquad y' = x - y^2.$$

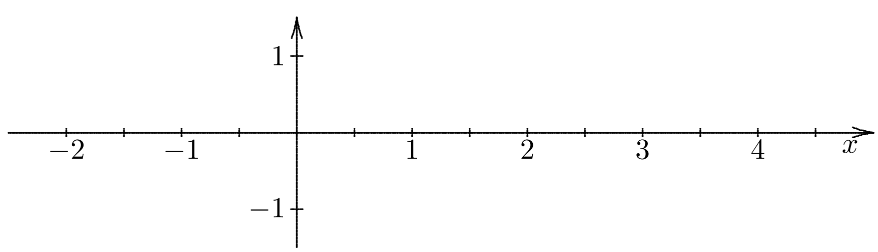
(a) Determine its singular (equilibrium) points find the linearized system near each. For each singular point, determine the type of its linearized behavior (focus, node, etc.) and whether is unstable, stable, or asymptotically stable for the linearized problem.

(b) Answer the same questions—type, stability—about the nature of these singular points in the true system, if you have the information to do so, or explain why you cannot.

3. The function $f(x)$ defined for $0 \le x \le 1$ by

$$f(x) = \begin{cases} x, & \text{if } 0 \le x \le 1/3, \\ 1, & \text{if } 1/3 < x \le 1 \end{cases}$$

has a quarter-range cosine series on this interval. (**You do not have to give this series or compute its coefficients.**) The series actually converges for all x to a certain extension $g(x)$ of $f(x)$. Plot $g(x)$ on the axes given, showing several periods. Don't worry here about values of $g(x)$ at points where it is discontinuous.



(c) To what value does the Fourier series in (a) actually converge (i.e., what is its sum) when

(i) $x = 0$? (ii) $x = 1/3$? (iii) $x = 3$?

4. Determine the equation of the phase trajectories of the system

$$x' = xy, \qquad y' = x^2$$

and sketch the phase plane with enough trajectories to show the essential features. Use arrows to indicate the direction of motion along the trajectories.

5. Consider the initial/boundary value problem

$$\begin{aligned} 9u_{xx} &= u_t, \quad 0 < x < 4,\ t > 0; \\ u_x(0,t) = 0, &\quad u_x(4,t) = 0, \quad t > 0; \\ u(x,0) = f(x) &= \begin{cases} 1, & \text{if } 0 \le x \le 2, \\ 0, & \text{if } 2 < x \le 4. \end{cases} \end{aligned}$$

(a) What sort of Fourier series (HRC, HRS, QRC, QRS, or other) is appropriate for this problem? Explain briefly.

(b) Find explicitly the expansion of $f(x)$ in the series you chose in (a). Simplify the coefficients to the extent possible.

(c) Give the solution $u(x,t)$ to the problem (as an infinite series). You are *not* required to derive the form of the solution.

6. Consider the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad 0 < x < 2; \qquad y'(0) = 0, \quad y(2) + y'(2) = 0.$$

You may assume (it is true) that all eigenvalues of this problem are (strictly) positive.

(a) Find an equation which λ must satisfy to be an eigenvalue, and for λ satisfying this condition write down an associated eigenfunction.

(b) There are an infinite number of positive eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$. Give a graphical construction of these, and from this find a good approximation for λ_n when n is very large. Label your graph clearly: axes, curves, etc.