section on the one hand and to learn how to use such software as well. To illustrate, let us use the *Maple* dsolve command (discussed at the end of Section 4.2) to obtain a Frobenius-type solution of the differential equation $xy'' + y = 0$ about the regular singular point $x = 0$; this was our Example 6. Enter

$$\text{dsolve}(x * \text{ diff}(y(x),\, x,\, x) + y(x) = 0,\ y(x),\ \text{type} = \text{series});$$

and return. The resulting output

$$\begin{aligned} y(x) = \_C1\, x \left(1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{144}x^3 + \frac{1}{2880}x^4 - \frac{1}{86400}x^5 + O(x^6)\right) \\ + \_C2 \left[\ln(x) \left(-x + \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{144}x^4 - \frac{1}{2880}x^5 + O(x^6)\right)\right. \\ \left. + \left(1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \frac{101}{86400}x^5 + O(x^6)\right)\right] \end{aligned}$$

is found to agree with the general solution that we generated in Example 5.

EXERCISES 4.3

1. For each equation, identify all singular points (if any), and classify each as regular or irregular. For each regular singular point use Theorem 4.3.1 to determine the minimum possible radii of convergence of the series that will result in (40) and (41) (but you need not work out those series).

(a) $y'' - x^3y' + xy = 0$
(b) $xy'' - (\cos x)y' + 5y = 0$
(c) $(x^2 - 3)y'' - y = 0$
(d) $x(x^2 + 3)y'' + y = 0$
(e) $(x + 1)^2y'' - 4y' + (x + 1)y = 0$
(f) $y'' + (\ln x)y' + 2y = 0$
(g) $(x - 1)(x + 3)^2y'' + y' + y = 0$
(h) $xy'' + (\sin x)y' - (\cos x)y = 0$
(i) $x(x^4 + 2)y'' + y = 0$
(j) $(x^4 - 1)y'' + xy' - x^2y = 0$
(k) $(x^4 - 1)^3y'' + (x^2 - 1)^2y' - y = 0$
(l) $(x^4 - 1)^3y'' - 3(x + 1)^2y' + x(x + 1)y = 0$
(m) $(xy')' - 5y = 0$
(n) $\left[x^3(x - 1)y'\right]' + 2y = 0$
(o) $2x^2y'' - xy' + 7y = 0$
(p) $xy'' + 4y' = 0$
(q) $x^2y'' - 3y = 0$
(r) $2x^2y'' + \sqrt{\pi}\, y = 0$

2. Sometimes one can change an irregular singular point to a regular singular point, by suitable change of variables, so that the Frobenius theory can be applied. The purpose of this exercise is to present such a case. We noted, in Example 3, that $y'' + \sqrt{x}\, y = 0$ $(x > 0)$ has an irregular singular point at $x = 0$, because of the $\sqrt{x}$.

(a) Show that if we change the independent variable from x to t, say, according to $\sqrt{x} = t$, then the equation on $y(x(t)) = Y(t)$ is

$$Y''(t) - \frac{1}{t}Y'(t) + 4t^3Y(t) = 0. \qquad (t > 0) \qquad (2.1)$$

(b) Show that (2.1) has a regular singular point at $t = 0$ (which point corresponds to $x = 0$).
(c) Obtain a general solution of (2.1) by the Frobenius method. (If possible, give the general term of any series obtained.) Putting $t = \sqrt{x}$ in that result, obtain the corresponding general solution of $y'' + \sqrt{x}\, y = 0$. Is that general solution for $y(x)$ of Frobenius form? Explain.
(d) Use computer software to find a general solution.

3. In each case, there is a regular singular point at the left end of the stated x interval; call that point x_0. Merely introduce a change of independent variable, from x to t, according to $x - x_0 = t$, and obtain the new differential equation on $y(x(t)) = Y(t)$. You need not solve that equation.

(a) $(x-1)y'' + y' - y = 0, \quad (1 < x < \infty)$
(b) $(x^2-1)y'' + y = 0, \quad (1 < x < \infty)$
(c) $(x+3)y'' - 2(x+3)y' - 4y = 0, \quad (-3 < x < \infty)$
(d) $(x-5)^2y'' + 2(x-5)y' - y = 0, \quad (5 < x < \infty)$

4. Derive the series solution (25).

5. Make up a differential equation that will have as the roots of its indicial equation

(a) $1, 4$ (b) $3, 3$ (c) $1/2, 2$ (d) $-1/2, 1/2$
(e) $2 \pm 3i$ (f) $-1, -1$ (g) $-2/3, 5$ (h) $-1 \pm i$
(i) $(1 \pm 2i)/3$ (j) $5/4, 8/3$

6. In each case verify that $x = 0$ is a regular singular point, and use the method of Frobenius to obtain a general solution $y(x) = Ay_1(x) + By_2(x)$ of the given differential equation, on the interval $0 < x < \infty$. That is, determine $y_1(x)$ and $y_2(x)$. On what interval can you be certain that your solution is valid? HINT: See Theorem 4.3.1.

(a) $2xy'' + y' + x^3y = 0$
(b) $xy'' + y' - xy = 0$
(c) $xy'' + y' + x^8y = 0$
(d) $xy'' + y' + xy = 0$
(e) $x^2y'' + xy' - y = 0$
(f) $x^2y'' - x^2y' - 2y = 0$
(g) $x^2y'' + xy' - (1+2x)y = 0$
(h) $x^2y'' + xy' - y = 0$
(i) $xy'' + xy' + (1+x)y = 0$
(j) $3xy'' + y' + y = 0$
(k) $x(1+x)y'' + y = 0$
(l) $x^2(2+x)y'' - y = 0$
(m) $x^2y'' - (2+3x)y = 0$
(n) $5xy'' + y' + 8x^2y = 0$
(o) $xy'' + e^xy = 0$
(p) $2xy'' + e^xy' + y = 0$
(q) $16x^2y'' + 8xy' - 3y = 0$
(r) $16x^2y'' + 8xy' - (3+x)y = 0$
(s) $x^2y'' + xy' + (\sin x)y = 0$
(t) $5(xy)'' - 9y' + xy = 0$
(u) $(xy')' - y = 0$
(v) $(xy')' - 2y' - y = 0$

7. (a)–(x) Use computer software to obtain a general solution of the corresponding differential equation in Exercise 6.

8. Use the method of Frobenius to obtain a general solution to the equation $xy'' + cy' = 0$ on $x > 0$, where c is a real constant. You may need to treat different cases, depending upon c.

9. (a) The equation

$$(x^2 - x)y'' + (4x-2)y' + 2y = 0, \quad (0 < x < 1) \tag{9.1}$$

has been "rigged" to have, as solutions, $1/x$ and $1/(1-x)$. Solve (9.1) by the method of Frobenius, and show that you do indeed obtain those two solutions.
(b) You may have wondered how we made up the equation (9.1) so as to have the two desired solutions. Here, we ask you to make up a linear homogenous second-order differential equation that has two prescribed LI solutions $F(x)$ and $G(x)$.

10. (*Complex roots*) Since $p(x)$ and $q(x)$ are real-valued functions, p_0 and q_0 are real. Thus, if the indicial equation (38) has complex roots they will be complex conjugates, $r = \alpha \pm i\beta$, so case (i) of Theorem 3.4.1 applies, and the method of Frobenius will give a general solution of the form

$$\begin{aligned} y(x) &= Ay_1(x) + By_2(x) \\ &= Ax^{\alpha+i\beta}\textstyle\sum_0^\infty a_nx^n + Bx^{\alpha-i\beta}\sum_0^\infty b_nx^n. \end{aligned} \tag{10.1}$$

(a) Show that the b_n's will be the complex conjugates of the a_n's: $b_n = \overline{a}_n$.
(b) Recalling, from Section 3.6.1, that

$$x^{\alpha \pm i\beta} = x^\alpha \left[\cos(\beta \ln x) \pm i \sin(\beta \ln x)\right], \tag{10.2}$$

show that (10.1) [with b_n replaced by $\overline{a}_n$, according to the result found in part (a) above] can be re-expressed in terms of real functions as

$$\begin{aligned} y(x) = Cx^\alpha &\left[\cos(\beta\ln x)\textstyle\sum_0^\infty c_nx^n \right. \\ &\left. - \sin(\beta \ln x)\textstyle\sum_0^\infty d_nx^n\right] \\ +Dx^\alpha &\left[\cos(\beta\ln x)\textstyle\sum_0^\infty d_nx^n \right. \\ &\left. + \sin(\beta \ln x)\textstyle\sum_0^\infty c_nx^n\right], \end{aligned} \tag{10.3}$$

where c_n, d_n are the real and imaginary parts of a_n, respectively: $a_n = c_n + id_n$.
(c) Find a general solution of the form (10.3) for the equation

$$x^2y'' + x(1+x)y' + y = 0.$$

That is, determine α, β and c_n, d_n in (10.3), through $n = 3$, say.
(d) The same as (c), for $x^2y'' + xy' + (1-x)y = 0$.

and return. Here, the initial conditions merely serve to establish the point about which the expansion is desired. The result is

$$y(x) = a + b(x-4) - \frac{1}{2}a(x-4)^2 - \frac{1}{6}b(x-4)^3 + \frac{1}{24}a(x-4)^4$$
$$+\frac{1}{120}b(x-4)^5 - \frac{1}{720}a(x-4)^6 - \frac{1}{5040}b(x-4)^7 + O\left((x-4)^8\right)$$

EXERCISES 4.2

1. Use (7a) or (7b) to determine the radius of convergence R of the given power series.

(a) $\sum_{0}^{\infty} nx^n$
(b) $\sum_{0}^{\infty}(-1)^n n^{1000} x^n$
(c) $\sum_{5}^{\infty} e^n x^n$
(d) $\sum_{0}^{1000} n!\, x^n$
(e) $\sum_{0}^{\infty}\left(\frac{x+3}{2}\right)^n$
(f) $\sum_{2}^{\infty}(n-1)^3(x-5)^n$
(g) $\sum_{1}^{\infty}\frac{n^{50}}{n!}(x+7)^n$
(h) $\sum_{2}^{\infty}(\ln n)^{n+1}(x-2)^n$
(i) $\sum_{3}^{\infty}\frac{(-1)^n}{4^n}(x+2)^{3n}$
(j) $\sum_{0}^{\infty}\frac{n}{2^n}(x-5)^{2n}$
(k) $\sum_{0}^{\infty}\frac{n^6}{3^n+n}(x+4)^{8n+1}$
(l) $\sum_{0}^{\infty}\frac{(-1)^{n+1}}{4^n+1}(x-3)^{2n+1}$

2. Determine the radius of convergence R of the Taylor series expansion of the given rational function, about the specified point x_0, using the ideas given in the paragraph preceding Example 5. Also, prepare a sketch analogous to those in Fig. 3.

(a) $\frac{1}{x^2+1}, \quad x_0 = 0$
(b) $\frac{1}{x^2+9}, \quad x_0 = 2$
(c) $\frac{x^3-2x+1}{x+7}, \quad x_0 = 5$
(d) $\frac{x+1}{x+2}, \quad x_0 = -26$
(e) $\frac{(x+1)^5}{x^2+3x+2}, \quad x_0 = -4$
(f) $\frac{x^2-3x+1}{x^2+2x+4}, \quad x_0 = 3$
(g) $\frac{x^2+x-2}{x^3-x^2+4x-4}, \quad x_0 = 2$
(h) $\frac{x^2-3x+2}{x-1}, \quad x_0 = 0$

3. Work out the Taylor series of the given function, about the given point x_0, and use (7a) or (7b) to determine its radius of convergence.

(a) $e^x, \quad x_0 = 1$
(b) $e^{-x}, \quad x_0 = -2$
(c) $\sin x, \quad x_0 = \pi$
(d) $\sin x, \quad x_0 = \pi/2$
(e) $\cos x, \quad x_0 = \pi/2$
(f) $\cos x, \quad x_0 = \pi$
(g) $\cos x, \quad x_0 = 5$
(h) $\ln x, \quad x_0 = 1$
(i) $x^2, \quad x_0 = 3$
(j) $2x^3 - 4, \quad x_0 = 0$
(k) $\cos(x-2), \quad x_0 = 2$
(l) $\frac{1}{1-x^{10}}, \quad x_0 = 0$
(m) $\frac{x^2}{1+x^{18}}, \quad x_0 = 0$
(n) $\sin(3x^{10}), \quad x_0 = 0$

4. Use computer software to obtain the first 12 nonzero terms in the Taylor series expansion of the given function f, about the given point x_0, and obtain a computer plot of f and the partial sums $s_3(x)$, $s_6(x)$, $s_9(x)$, and $s_{12}(x)$ over the given interval I.

(a) $f(x) = e^{-x}, \quad x_0 = 0, \quad I:\ 0 < x < 4$
(b) $f(x) = \sin x, \quad x_0 = 0, \quad I:\ 0 < x < 10$
(c) $f(x) = \ln x, \quad x_0 = 1, \quad I:\ 0 < x < 2$
(d) $f(x) = 1/(1-x), \quad x_0 = 0, \quad I:\ -1 < x < 1$
(e) $f(x) = 1/x, \quad x_0 = 2, \quad I:\ 0 < x < 4$
(f) $f(x) = 1/(1+x^2), \quad x_0 = 0, \quad I:\ -1 < x < 1$
(g) $f(x) = 4/(4+x+x^2), \quad x_0 = 0, \quad I:\ -1.3 < x < 0.36$

5. (*Geometric series*) (a) Show that

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^{n-1} + \frac{x^n}{1-x}} \qquad (5.1)$$

is an *identity* for all $x \neq 1$ and any positive integer n, by multiplying through by $1-x$ (which is nonzero since $x \neq 1$) and simplifying.
(b) The identity (5.1) can be used to study the Taylor series

known as the **geometric series** $\sum_{k=0}^{\infty} x^k$ since, according to (5.1), its partial sum $s_n(x)$ is

$$s_n(x) = \sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}. \qquad (x \neq 1) \tag{5.2}$$

Show, from (5.2), that the sequence $s_n(x)$ converges, as $n \to \infty$, for $|x| < 1$, and diverges for $|x| > 1$.
(c) Determine, by any means, the convergence or divergence of the geometric series for the points at the ends of the interval of convergence, $x = \pm 1$. NOTE: The formula (5.2) is quite striking because it reduces $s_n(x)$ to the *closed form* $(1 - x^n)/(1 - x)$, direct examination of which gives not only the interval of convergence but also the sum function $1/(1-x)$. It is rare that one can reduce $s_n(x)$ to closed form.

6. (a) Derive the Taylor series of $1/(x-1)$ about $x = 4$ using the Taylor series formula (16), and show that your result agrees with (38).
(b) Show that the same result is obtained (more readily) by writing

$$\frac{1}{x-1} = \frac{1}{3+(x-4)} = \frac{1}{3}\frac{1}{1+\frac{x-4}{3}} \tag{6.1}$$

and using the geometric series formula

$$\boxed{\frac{1}{1-t} = \sum_{0}^{\infty} t^n \qquad (|t| < 1)} \tag{6.2}$$

from Exercise 5, with $t = -(x-4)/3$. Further, deduce the x interval of convergence of the result from the convergence condition $|t| < 1$ in (6.2).

7. For each of the following differential equations do the following: Identify $p(x)$ and $q(x)$ and, from them, determine the least possible guaranteed radius of convergence of power series solutions about the specified point x_0; seeking a power series solution form, about that point, obtain the recursion formula and the first four nonvanishing terms in the power series for $y_1(x)$ and $y_2(x)$; verify that y_1, y_2 are LI.

(a) $y'' + 2y' + y = 0, \quad x_0 = 0$
(b) $y'' + 2y' = 0, \quad x_0 = 0$
(c) $y'' + 2y' = 0, \quad x_0 = 3$
(d) $xy'' + y' + y = 0, \quad x_0 = -5$
(e) $xy'' - 2y' + xy = 0, \quad x_0 = 1$
(f) $x^2y'' - y = 0, \quad x_0 = 2$
(g) $xy'' + (3+x)y' + xy = 0, \quad x_0 = -3$
(h) $y'' + y' + (1 + x + x^2)y = 0, \quad x_0 = 0$
(i) $y'' - (1+x^2)y = 0, \quad x_0 = 0$
(j) $y'' - x^3y = 0, \quad x_0 = 0$
(k) $y'' + x^3y' + y = 0, \quad x_0 = 0$
(l) $y'' + xy' + x^2y = 0, \quad x_0 = 0$
(m) $y'' + (x-1)^2y = 0, \quad x_0 = 2$

8. (a)–(m) Use computer software to obtain the general solution, in power series form, for the corresponding problem given in Exercise 7, about the given expansion point.

9. (*Airy equation*) For the **Airy equation**,

$$y'' - xy = 0, \qquad (-\infty < x < \infty) \tag{9.1}$$

derive the power series solution

$$\begin{aligned} y(x) &= a_0 y_1(x) + a_1 y_2(x) \\ &= a_0\left(1 + \sum_{1}^{\infty} \frac{x^{3n}}{2\cdot 3\cdots(3n-1)(3n)}\right) \\ &\quad + a_1\left(x + \sum_{1}^{\infty} \frac{x^{3n+1}}{3\cdot 4\cdots(3n)(3n+1)}\right) \end{aligned} \tag{9.2}$$

and verify that it is a general solution. NOTE: These series are not summable in closed form in terms of elementary functions thus, certain linear combinations of y_1 and y_2 are adopted as a usable pair of LI solutions. In particular, it turns out to be convenient (for reasons that are not obvious) to use the **Airy functions** $Ai(x)$ and $Bi(x)$, which satisfy these initial conditions: $Ai(0) = 0.35502$, $Ai'(0) = -0.25881$ and $Bi(0) = 0.61493$, $Bi'(0) = 0.44829$.

10. Use computer software to obtain power series solutions of the following initial-value problems, each defined on $0 \le x < \infty$, through terms of eighth order, and obtain a computer plot of $s_2(x)$, $s_4(x)$, $s_6(x)$, and $s_8(x)$.

(a) $y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$
(b) $y'' + x^2y = 0, \quad y(0) = 2, \quad y'(0) = 0$
(c) $y'' - xy' + y = 0, \quad y(0) = 0, \quad y'(0) = 1$
(d) $(1+x)y'' + y = 0, \quad y(0) = 2, \quad y'(0) = 0$
(e) $(3+x)y'' + y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1$
(f) $(1+x^2)y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 1$

11. From the given recursion formula alone, determine the radius of convergence of the corresponding power series solutions.

(a) $(n+3)(n+2)a_{n+2} - (n+1)^2a_{n+1} + na_n = 0$
(b) $(n+1)a_{n+2} + 5na_{n+1} + a_n - a_{n-1} = 0$
(c) $(n+1)^2a_{n+2} + (2n^2+1)a_{n+1} - 4a_n = 0$
(d) $(n+1)a_{n+2} - 3(n+2)a_n = 0$
(e) $na_{n+2} + 4na_{n+1} + 3a_n = 0$
(f) $n^2a_{n+2} - 3(n+2)^2a_{n+1} + 3a_{n-1} = 0$

12. In the Comment at the end of Example 6 we wondered what the divergence of the series solution over $7 < x < \infty$

implied about the nature of the solution over that part of the domain. To gain insight, we propose studying a simple problem with similar features. Specifically, consider the problem

$$(x-1)y' + y = 0, \qquad y(4) = 5 \tag{12.1}$$

on the interval $4 \leq x < \infty$.
(a) Solve (12.1) analytically, and show that the solution is

$$y(x) = \frac{15}{x-1} \tag{12.2}$$

over $4 \leq x < \infty$. Sketch the graph of (12.2), showing it as a solid curve over the domain $4 \leq x < \infty$, and dotted over $-\infty < x < 4$.
(b) Solve (12.1), instead, by seeking $y(x) = \sum_0^\infty a_n(x-4)^n$.
(c) Show that the solution obtained in (b) is, in fact, the Taylor expansion of (12.2) about $x = 4$ and that it converges only in $|x-4| < 3$ so that it represents the solution (12.2) only over the $4 \leq x < 7$ part of the domain, even though the solution (12.2) exists and is perfectly well-behaved over $7 < x < \infty$.

13. Rework Example 5 without using the $\sum$ summation notation. That is, just write out the series, as we did in the introductory example of Section 4.1. Keep powers of x up to and including fifth order, x^5, and show that your result agrees (up to terms of fifth order) with that given in (29).

14. Rework Example 6 without using the $\sum$ summation notation. That is, just write out the series as we did in the introductory example of Section 4.1. Keep powers of $x - 4$ up to and including fourth order $(x-4)^4$, and show that your result agrees (up to terms of fourth order) with that given in (46).

15. (*Cesàro summability*) Although (5) gives the usual definition of infinite series, it is not the only possible one nor the only one used. For example, according to **Cesàro summability**, which is especially useful in the theory of Fourier series, one defines

$$\sum_1^\infty a_n \equiv \lim_{N\to\infty} \frac{s_1 + s_2 + \cdots + s_N}{N}, \tag{15.1}$$

that is, the limit of the arithmetic means of the partial sums. It can be shown that if a series converges to s according to "ordinary convergence" [equation (5)], then it will also converge to the same value in the Cesàro sense. Yet, there are series that diverge in the ordinary sense but that converge in the Cesàro sense. Show that for the geometric series (see Exercise 5),

$$\frac{s_1 + s_2 + \cdots + s_N}{N} = \frac{1}{1-x} - \frac{x}{N}\frac{1-x^N}{(1-x)^2} \tag{15.2}$$

for all $x \neq 1$, and use that result to show that the Cesáro definition gives divergence for all $|x| > 1$ and for $x = 1$, and convergence for $|x| < 1$, as does ordinary convergence, but that for $x = -1$ it gives convergence to $1/2$, whereas according to ordinary convergence the series diverges for $x = -1$.

4.3 The Method of Frobenius

4.3.1. Singular points. In this section we continue to consider series solutions of the equation

$$y'' + p(x)y' + q(x)y = 0. \tag{1}$$

From Section 4.2, we know that we can find two LI solutions as power series expansions about any point x_0 at which both p and q are analytic. We call such a point x_0 an **ordinary point** of the equation (1). Typically, p and q are analytic everywhere on the x axis except perhaps at one or more singular points, so that all points of the x axis, except perhaps a few, are ordinary points. In that case one can readily select such an x_0 and develop two LI power series solutions about that point.

Nevertheless, in the present section we examine singular points more closely, and show that one can often obtain modified series solutions about singular points.