MATH 640:491–Fall, 2006

Suggested problems for 10/3/06[1]

1. Let $T_0 = 2, T_1 = 3$ and $T_2 = 6$. For $n \geq 3$, let $T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}$. The first few terms are:

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$ where A_n and B_n are well known sequences.

(Putnam 1990)

2. Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$

Prove that there is a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

(Putnam 1993)

3. Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all integers $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

(Putnam 1997)

4. Let S_0 be a finite set of positive integers. We define sets $S_1, S_2, \ldots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a-1$ and a is in S_n. Show that there exist infinitely many integers N such that $S_N = S_0 \cup \{N + a : a \in S_0\}$.

(Putnam 2000)

5. Evaluate:

$$\sum_{i=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}.$$

(Putnam exam 1984)

6. Say that a set of integers is *selfish* if it has its own cardinality (number of elements) is a member. Find, with proof, the number of subsets of $\{1, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish. (Putnam exam, 1996)

[1] Version:9/26/06

7. Given a finite string S of symbols X and O, we write $\Delta(S)$ for the number of X's minus the number of O's of S. We call a string S *balanced* if every substring of T of consecutive symbols of S satisfies $-2 \le \Delta(T) \le 2$. Find with proof, the number of balanced strings of length n.

   (Putnam 1996)

8. For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$ and for $k \ge 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

   Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, for $1 \le k \le n$. (Putnam exam 1997)

9. Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansion of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2A_1 = 10$, and $A_4 = A_3A_2 = 101$. Determine all n such that 11 divides A_n. (Putnam