Mathematics 452 Abstract Algebra II Spring 2015

EXAMINATION II- Take Home Exam- Version 4/25/15 9:50 AM

You may consult any written references that you wish. Do not discuss the exam with anyone except the instructor. You may quote any theorems from the book or class lectures. All proofs should be carefully explained. Turn in your solutions to the problems by 4 PM on 5/8/14 on your own paper, making sure your name and RUID are on the pages.

Be sure to check back on the website to see if any
misprints or changes have been made to the take home exam.
A note will be posted on the assignments page if changes are posted

Show and explain all of your work

1.(15 points) Recall that square matrices C, D with entries in a field F are similar if there exists an invertible matrix P with entries in F such that $P^{-1}CP = D$.

a) Let C, D be $n \times n$ matrices with rational entries. Suppose that there exists an invertible complex matrix P such that $P^{-1}CP = D$. Show that there exists a matrix R with rational coefficients which is invertible such that $R^{-1}CR = D$.

b) Arrange the rational matrices below into similarity classes (be sure to explain precisely why matrices are or are not similar):

$$\begin{pmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 2 & -1 \\ 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & -3 \\ -1 & 0 & 1 & -4 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 4 \end{pmatrix}$$

2.(20 points) (This problem is related to Artin 12.M.7b). Let R be a unique factorization domain such that for each irreducible element $\pi \in R$ the quotient ring $R/\pi R$ is a finite ring.

a) Give an example of a UFD satisfying the conditions above and a UFD that does not satisfy the conditions above.

b) For each nonzero $r \in R$ show that there are R submodules $(r) = M_0 \subset M_1 \subset M_2 \subset \cdots M_k = R$ such that M_i/M_{i-1} is a simple module. Show that $R/(r)$ is a finite ring for any nonzero $r \in R$.

c) Assume for the remainder of the problem that $f(x)$ and $g(x)$ are polynomials in $R[x]$ such that the only common divisors are units in $R[x]$. Let $\pi \in R$ be irreducible, and let j be a positive integer. Show that the ideal J generated by the images of $f(x), g(x)$ in $R/(\pi^j)[x]$ contains a polynomial of the form $m(x)+\pi^i h(x)$ for some $i \geq 1$, where $m(x) \in R/(\pi^j)[x]$ is monic.

d) With the notation of part d), let p be the characteristic of $R/(\pi)$. By repeatedly raising to p-th powers the polynomial found in part d) show that the ideal $J \subset R/(\pi^j)[x]$ contains a monic polynomial.

e) Show that there is a nonzero element of R in the ideal $(f(x), g(x)) \subset R[x]$.

f) Use the previous parts to show that R[x]/(f,g) is a finite ring (thus solving Artin 12.M.7).

3.(20 points) Let F be a field and let $R = F[x]$ be the ring of polynomials with coefficients in F.

a) Consider $F[x]$-modules $M = F[x]/(m(x)), N = F[x]/(n(x))$ for some nonzero polynomials $m(x), n(x) \in F[x]$. Show that the set $Hom_{F[x]}(M, N)$ of $F[x]$ module homomorphisms from M to N is a finite dimensional F vector space.

b) Show that an element of $Hom_{F[x]}(M, N)$ is determined by the image of a generator of the cyclic module M, and that this image can be chosen to be any element of the set $\{z \in N | m(x)z = 0\}$. Determine the dimension over F of $Hom_{F[x]}(M, N)$ when $m(x)|n(x)$ in terms of degrees of the polynomials. Also find the dimension when $n(x)|m(x)$.

c) Suppose that W is a finite sum of cyclic $F[x]$ modules $F[x]/(d_i(x))$ for some nonzero polynomials $d_i(x) \in F[x]$ such that $d_i(x)|d_{i+1}(x)$. Find a formula for the dimension of $Hom_{F[x]}(W, W)$

d) Let V be a k-dimensional vector space over F and let L be a linear transformation of V to itself represented by a matrix C with respect to some basis. Compute the dimension of the set of all $k \times k$ matrices which commute with C by using L to consider V as an $F[x]$-module and applying the previous parts of the problem.

4.(15 points) (This problem is related to Artin 12.M.11) Let F be a finite field with 2 elements and let $f(x) \in F[x]$ be a polynomial of degree n. Complete the steps below to show that factoring a polynomial over a finite field can be reduced to the relatively fast operations of linear algebra and Euclidean algorithm.

a) Show that if $f(x)$ is reducible and $f(x), f'(x)$ are relatively prime then the ring $F[x]/(f(x))$ contains an idempotent element g which is not 0 or 1.

b) Show that finding all idempotents in $F[x]/(f(x))$ can be done by solving n linear equations with coefficients in the field F.

c) Show that if g is a nontrivial idempotent as in a), then one of $\gcd(f, g), \gcd(f, g-1)$ is a proper factor of f(x).

d) Factor the polynomial $x^9 + x^6 + x^4 + 1 \in F_2[x]$ into a product of irreducible factors. Be sure to prove that your factors are irreducible.

5.(14 points) Artin 15.M.7

6.(15 points) (This problem is related to Artin 15.M.5) Let A be an invertible $n \times n$ matrix with rational entries.

a) Show that a complex eigenvalue of A is contained in a field L with the degree of $L/\mathbf{Q}$ at most n.

b) Show that if $n \leq 3$ and A has finite order in the group of invertible matrices, then it has order 1,2,3,4 or 6.

c) Find a square matrix of order 5 in the group of $n \times n$ invertible rational matrices for each integer $n \geq 4$.