Mathematics 421 Essay 4
Partial Differential Equations
Spring 2010

0. Introduction

The **Method of Separation of Variables** is used to construct special solutions of some linear partial differential equations. Other solutions can then be found as linear combinations of these special solutions. Initially, the boundary conditions will be homogeneous, and our applications will use **eigenfunction expansions** of initial conditions to produce an expression for the solution of equations with the given boundary conditions. Since the equations are **linear** the difference of two solutions of **the same** equation with **the same** initial and boundary conditions will be the solution of that equation with **zero** initial and boundary conditions. The **uniqueness** question for the equation is then equivalent to the statement that the only solution with zero initial and boundary is the zero function. We only give a physical justification of this, not a mathematical proof. On the other hand, our method of solution gives an affirmative answer to the **existence** question by showing that the equation has a solution, at least on special regions, for fairly general initial conditions.

1. Some partial differential equations

Equations involving **partial derivatives** arise in certain physical models. The model gives rise to the equation and certain initial conditions (in time) and boundary conditions (in space). The combined problem is interesting because of its relation to the physical model. The discussion of the model determines the conditions under which we can expect unique solutions as well as a justification that this our method identifies the **physically correct** solution. In addition to describing the method, we shall give an outline of a method for verifying its validity. The form of the solution involves a series whose convergence is not assured, so more work must be done to even show that it satisfies the equation. The equations for $u(x,t)$ that we study, together with additional conditions are the **heat equation**:

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad (k > 0); \tag{H}$$

and the **wave equation**

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}. \tag{W}$$

In both, there will be **boundary conditions** for x and **initial conditions** for t. The equations of second order in x, so boundary conditions at two points are suitable according to the theory developed in the notes on **Boundary Value Problems** (Essay 3). Since (H) is a first order equation with respect to t, the value of $u(x,0)$ should suffice to determine a solution; but (W) is a second order equation with respect to t, so the value of u and $\partial u/\partial t$ at $t = 0$ should be specified. Further justification of these assumptions will be given in later sections.

Note on notation: The use of expressions like $\partial u/\partial t$ is awkward and usually requires even simple equations to be displayed. A useful alternative that allows evaluation of derivatives at special values of the variables is to denote the derivative with respect a particular variable by writing that variable as a **subscript** of the function name. This will be used generally in lecture, quizzes, and exams.

In the examples in this supplement, we will take the boundary conditions to be $u(0,t) = u(L,t) = 0$. Once you are familiar with the method, the changes necessary to deal with other boundary conditions will be easy to implement. Different boundary conditions lead to different eigenfunctions, so the special features of the selected boundary value problems must be considered before starting to compute the solution.

Some boundary conditions (including the one we have chosen) lead to something resembling the **half-range Fourier series**. Although the functions are determined by their properties on the interval $[0,L]$, they have extensions that are periodic with period $2L$. If the series can be identified with a Fourier series, then known Fourier series and operational properties of Fourier series can be used to obtain the eigenfunction expansion of the initial values that are used to give an expression for the solution of the given equation.

2. Separation of variables for the heat equation

The equations (H) and (W) are **linear** and **homogeneous**, so **linear combinations** of solutions are also solutions. The method of solution will be to attempt to express the desired solution in terms of special solutions of the equation together with **boundary conditions**, using the **initial conditions** to determine the coefficients. This is similar to the **power series** method for solving initial value problems for ordinary differential equations. In that method, a **recurrence** is used to express all coefficients of the general solution in terms of the coefficients of terms of lowest degree. Then, Taylor's formula is used to relate those coefficients to the values of the solution and its derivatives—the initial data—at $x = 0$.

The special solutions will be assumed to have the form $u(x,t) = X(x)T(t)$. For such expressions, (H) becomes

$$kX''(x)T(t) = X(x)T'(t) \quad \text{or} \quad \frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)}.$$

The second form asserts that a function of t alone is equal to a function of x alone. This is only possible is both functions are **constant**. We also must have the boundary condition $X(0) = X(L) = 0$. We have already seen that these boundary values, together with an equation saying that $X''(x)/X(x)$ is constant gives that $X(x) = \sin(n\pi x/L)$ for $n = 1, 2, \ldots$, For this choice of X, $X''(x)/X(x) = -n^2\pi^2/L^2$. Then, $T(t) = e^{-kn^2\pi^2 t/L^2}$. Combining these solutions for all n gives a solution of the differential equation and boundary values of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-kn^2\pi^2 t/L^2} \sin(n\pi x/L) \qquad (U_H),$$

but we have yet to satisfy the initial condition. When $t = 0$, the solution reduces to

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L).$$

If the c_n are chosen to give the **eigenfunction expansion** of the given value of $u(x,0)$, (U_H) **should be** an expression for the solution of (H). For these boundary conditions, the series is a half-range Fourier series of a function of period $2L$ found by extending the given $u(x,0)$ to an odd function on $[-L, L]$.

The text illustrates this for $k = 1$, $L = \pi$ and $u(x,0) = 100$, for which

$$c_n = \frac{2}{\pi}\int_0^{\pi} 100 \sin nx \, dx = \frac{200}{\pi}\left[\frac{-1}{n}\cos nx\right]_0^{\pi} = \frac{200(1-(-1)^n)}{n\pi}$$

The coefficients are $400/(n\pi)$ for odd n and zero for even n.

When $t = 0$ this expression has been shown to converge to the given initial value for all x with $0 < x < \pi$. The situation is better for $t > 0$ since c_n is multiplied by $e^{-kn^2\pi^2 t/L^2}$, which decreases rapidly as n increases. The resulting series converges uniformly for $t > 0$, justifying the formal manipulations in verifying the solution.

It is not surprising that the initial conditions should present difficulty since $u(0,0)$ is required to be both 0 and 100 depending on whether it is viewed as a specialization of the boundary condition at $x = 0$ or the initial condition at $t = 0$.

The textbook has some graphs of the temperature distribution as a function of x for several values of t and the temperature as a function of t for fixed x.

As a further illustration of the method, consider exercise 13.3.1 This has $0 \le x \le L$ with $u(0,t) = u(L,t) = 0$ and

$$u(x,0) = \begin{cases} 1 & \text{if } 0 < x < L/2 \\ 0 & \text{if } L/2 < x < L \end{cases}$$

The eigenfunctions are $\sin n\pi x/L$ and the corresponding $T(t)$ is $e^{-kn^2\pi^2 t/L}$ The expression for c_n is

$$c_n = \frac{2}{L}\int_0^{L/2} \sin n\pi x/L\,dx = \frac{2}{L}\left[\frac{-L}{n\pi}\cos n\pi x/L\right]_0^{L/2} = \frac{2}{n\pi}(1 - \cos n\pi/2)$$

From $n = 1$ to $n = 4$, the values of $1 - \cos n\pi/2$ are $1, 2, 1, 0$. Then these repeat. Since the initial value is discontinuous, c_n only goes to zero as $1/n$, but the series for $t > 0$ converges more rapidly, leading to a continuous $u(x,t)$ as in the first example.

Note that, when a function is defined by cases, the integrals defining the coefficients in the eigenfunction expansion are evaluated as a sum of **case integrals**, where the domain of integration is restricted to an interval where a particular formula represents the function and that formula is used in place of the function.

Note that k only appears as a factor multiplying t in the exponent of $T(t)$ and L only appears as a scale factor in expressions x/L and kt/L, but **not** in the expression for the value of c_n. The inclusion of these parameters **appears** more general, but they serve only to express the **units of measurement**.

Maple was used to find the sum of the terms from 1 to 100 (because of the distribution of coefficients, this is 75 terms) with $L = \pi$ and $k = 10^{-4}$. Figure 1 shows the graph for $t = 1$.

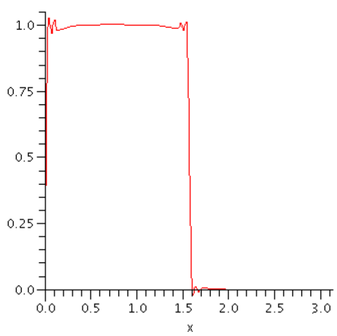

Figure 1

There is very little change from the original distribution: the oscillations in the graph are a consequence of ignoring terms that have a visible effect on the graphs. At $t = 200$, the graph of the series is shown in Figure 2.

The terms through $n = 100$ now suffice to give a smooth picture of $u(x, 200)$. There has been very little change in temperature at positions between 0.5 and 1.5. Otherwise there is a clear concave downward shape in the left side of the interval and concave upward in the right side.

By the time we reach $t = 20000$, the maximum temperature is less than $1/10$ of its original value and it resembles the **first term** of the series (although we still used the sum through $n = 100$). Figure 3 shows this graph, and the behavior of the temperature at $x = 2$ for t from 0 to 10000 is shown in Figure 4.

Note that the temperature increases until about $t = 3000$, and then starts to decrease.

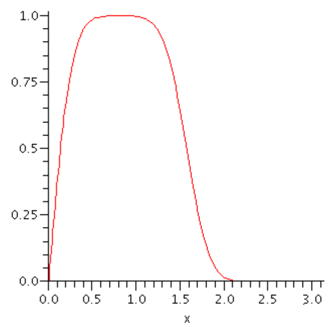

Figure 2

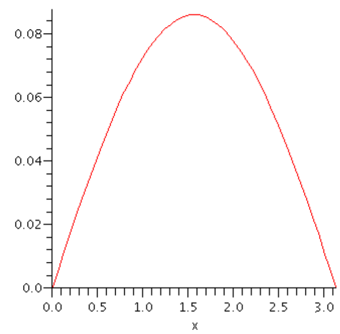

Figure 3

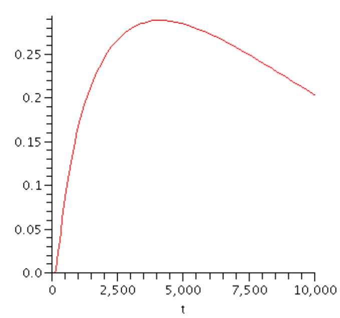

Figure 4

3. Separation of variables for the wave equation

Again, (W) is a linear equation, so linear combinations of solutions with homogeneous boundary conditions will satisfy (W) and the same boundary conditions. Thus, we try solutions of the form $u(x,t) = X(x)T(t)$ in the expectation that there will be enough solutions of this form to represent all initial conditions. The equation gives $a^2X''(x)T(t) = X(x)T''(t)$, and dividing by $X(x)T(t)$ gives a function of x equal to a function of t, which requires that the common value is constant. The boundary conditions have already been seen to allow nontrivial conditions only if $X''(x)/X(x) < 0$, so we write

$$\frac{X''}{X} = -\alpha^2 \quad \text{and} \quad \frac{T''}{T} = -a^2\alpha^2$$

Boundary conditions of the form $u(0,t) = u(L,t) = 0$ give $X(x) = \sin \alpha_n x$ with $\alpha_n = n\pi/L$. The corresponding $T(t)$ is $A_n \cos a\alpha_n t + B_n \sin a\alpha_n t$. Since there are **two** constants, A and B, to be determined, **two** equations will be needed. The second equation comes from consideration of $T'(t) = a\alpha_n(-A_n \sin a\alpha_n t + B_n \cos a\alpha_n t)$. The A_n and B_n will be chosen based on the initial values of $u(x,0)$ and $u_t(x,0)$. Putting $t = 0$ in these expressions gives

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \alpha_n x$$

$$u_t(x,0) = \sum_{n=1}^{\infty} a\alpha_n B_n \sin \alpha_n x$$

The A_n and B_n are coefficients in eigenfunction expansions, so we have

$$A_n = \frac{2}{L}\int_0^L u(x,0)\sin \alpha_n x\, dx$$

$$a\alpha_n B_n = \frac{2}{L}\int_0^L u_t(x,0)\sin \alpha_n x\, dx$$

In **plucked string** examples, there is an initial **displacement**, but zero initial **velocity**, so all $B_n = 0$. The mathematics of other examples is not significantly different, but this family of examples is easier to visualize, so most examples will have this form. In exercise 13.4.1, we assume that $u(x,0) = x(L-x)/4$. Note that this formula is only assumed valid for $0 \le x \le L$. Since the **eigenfunctions** of this **Sturm-Liouville** problem are the same as the **odd half-range Fourier series** of period $2L$, it is the odd periodic extension that is used to extend the initial condition to the whole real line. Ignoring the details of finding the indefinite integral by integration by parts, and using the special values $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, we have

$$A_n = \frac{L^2(1-(-1)^n)}{n^3\pi^3}$$

and the solution is

$$u(x,t) = \frac{L^2}{\pi^3}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \sin\frac{n\pi x}{L} \cos\frac{an\pi t}{L}$$

4. d'Alembert's solution of the wave equation

There is another approach to the one-dimensional wave equation that sometimes gives a closed form solution. Unfortunately, it fails to apply to other equations of interest in applications. However, it does illustrate our observation that partial differential equations have many solutions.

The key observation is that

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}\right) - a\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}\right).$$

This expression can be written in a simpler form using the **change of variables**

$$\xi = x + at$$

$$\eta = x - at$$

If u is considered as defined as a function of ξ and η that becomes a function of x and t by this expression, then the **chain rule** says that

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}\\ &= \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t}\\ &= a\frac{\partial u}{\partial \xi} - a\frac{\partial u}{\partial \eta}\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} &= 2a\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial t} - a\frac{\partial u}{\partial x} &= -2a\frac{\partial u}{\partial \eta}\end{aligned}$$

The functions $u(\xi, \eta)$ satisfying $u_t + au_x = 0$ are precisely those that are independent of ξ, and those with $u_t - au_x = 0$ are precisely those that are independent of η. Both families of functions satisfy the wave equation, and every solution is easily seen to be a sum of functions of these two types.

The only difficulty in this analysis is that the notation for partial derivatives is somewhat misleading. A partial derivative is not simply the derivative of one quantity with respect to another, but it depends on the context of the quantities. A single independent variable has no absolute meaning; it must be considered as a member of a **set** of variables that are used to describe other quantities. Because of this, it is best to avoid using similar names for members of different sets of variables that can be used as independent variables. When one pair of variables, like (x, y), is defined in terms of another, like (ξ, η), it is the **matrix** of partial derivatives to which the familiar tools of calculus apply, not the individual partial derivatives. In particular, the chain rule for functions of several variables has the form of **matrix multiplication**. The derivative of the inverse of a given function is given by inverse of the **matrix of partial derivatives** of the function. In this case, we did not actually construct this matrix to obtain quantities playing the role of derivatives with respect to ξ and η while expressed in terms of the partial derivatives with respect to x and y, but we did all the work necessary to invert the matrix. A more interesting case that we will meet later involves the relation between the rectangular (x, y) coordinates and polar coordinates (r, θ). Here, each point has **many** different descriptions in polar coordinates, so there is no single **function** giving the polar coordinates in terms of x and y (you may have seen a formula in your calculus book, but it is not defined **everywhere**). Nevertheless, it is possible to invert the matrix of partial derivatives to get the partial derivatives of r and θ with respect to x and y as functions of r and θ.

If $u = F(x + at) + G(x - at)$, then $u_t = a\big(F'(x + at) - G'(x - at)\big)$. Also, $u_x = F'(x + at) + G'(x - at)$. Thus, if we are given u and u_t as functions of x for $t = 0$, we can differentiate u with respect to x to get $u_x(x, 0)$. This gives both $F' + G'$ and $F' - G'$ as functions of x when $t = 0$. These can be combined to get $F'(x)$ and $G'(x)$ individually, and these integrated to give $F(x)$ and $G(x)$ individually (except for the constants of integration). Since F and G were given as functions of a single variable, expressions for those functions are now known (the reader is encouraged to fill in the details for dealing with the constants of integration).

This solution has a **visual** interpretation. $G(x - at)$ describes a function whose value at x when t is increased by h is the same as its current value ah units to the left. The wave is then seen as a **traveling wave** moving to the right at velocity a. Similarly, $F(x + at)$ represents something moving to the left at the same speed. The general solution of the equation is a **superposition** of these two solutions.

This is particularly useful then seeking solutions of the wave equation when x lies in an **infinite** interval, either the whole line, or the interval where $x > 0$, but boundary conditions can be translated into properties of an extension of the solution to the whole line, leading to the appearance of a traveling wave being **reflected** at the boundary.

5. Exercises

A. Suppose that we impose the **insulated end** condition, $u_x(0,t) = u_x(L,t) = 0$ in the heat equation (H). For product solutions, this requires $X'(0) = X'(L) = 0$. What are the eigenfunctions in this case? As a simple example, suppose that the initial condition is $u(x,0) = 100$. What is $u(x,t)$? Why is this answer **obvious**?

B. As in (A), impose the **insulated end** condition, $u_x(0,t) = u_x(L,t) = 0$ in the heat equation (H). This time, assume

$$u(x,0) = \begin{cases} 1 & \text{if } 0 < x < L/2 \\ 0 & \text{if } L/2 < x < L \end{cases}$$

Find the solution. If t is large, what is the temperature distribution?

C. Using **separation of variables**, we found solutions of $u_{tt} - a^2 u_{xx} = 0$ of the form

$$\sin cx \sin act \qquad \text{and} \qquad \sin cx \cos act.$$

Write these solutions in the form $F(x + at) + G(x - at)$ appearing in d'Alembert's solution.

End of supplement