Mathematics 421 Essay 2
An operational view of Fourier coefficients
Spring 2010

0. Introduction

Although Laplace transforms were defined using an integral containing a parameter, the computation of transforms and inverse transforms used a special list of properties. This approach is generally known as an "operational" approach. Techniques of integration appear only where **necessary** to derive the properties that are taken as the basic properties of the transform. Although the properties of the transform (except for linearity) are as exotic as the rules of calculus, they become easy to use with a little practice.

Unfortunately, the treatment of Fourier coefficients in the text doesn't develop a similar list of properties, relying instead on classical techniques of integration. Each exercise becomes a separate calculation and results that provide structural hints to the form of the answer are hidden. Since these separate calculations are frequently tedious, leading to computational mistakes, they are a poor substitute for methods that emphasize **visible** properties of correct answers.

There is some concern about the claim that the Fourier series is **equal** to the function it represents or its **periodic extension**. We expect equality to be preserved when we evaluate expressions at particular values of a variable in the equation. However, Fourier series don't always converge in this **pointwise** sense to the function that they represent. While this type of equality may fail, other properties are easily seen to be preserved. In particular, the integrals defining the coefficients behave in the expected way, so the function determines its Fourier series. This allows us to show that the coefficients do tend to zero and the rate at which they decrease is related to the smoothness of the function. This provides a visual clue to what to expect from a computation of Fourier coefficients that is useful in detecting serious mistakes. For functions that are **continuous** as periodic functions on the whole real line and **piecewise differentiable**, the series converges to the function uniformly. Functions arising applications are often this nice, so we may consider them as being equal to their Fourier series. In the more general case of function that are only piecewise continuous with jump discontinuities, the behavior of the series is well understood. At a jump, the series converges to the value exactly in the middle of the jump.

The ability to recover the function from its Fourier series gives an important check on the computations. For example, one operation that was useful in dealing with Laplace transfors was the **shifting** theorem for relating the transforms of $f(t)$ and $f(t-c)$. For Fourier series, the same proof may be used, but the conclusion is more natural: the Fourier series for $f(x-c)$ is obtained by substituting $x-c$ for x in the Fourier series for $f(x)$.

It is also true, to some extent, that term-by-term differentiation of the series corresponds to differentiation of the function. As with Laplace transforms, this is proved using integration by parts, but it is easier to remember that the series for $f'(x)$ is essentially the term by term derivative of the series for $f(x)$.

Part of the difficulty is that **real** Fourier series use trigonometric functions, so there are **two** functions for each positive index, and the term with index zero needs to be treated differently. The use of **complex exponentials** gives a single list of functions indexed by **all** integers, and the partial sums of the series are taken to be the sum of terms with index between $-n$ and $+n$ for integers n.

Since the Fourier coefficients are defined by integrating certain expressions, they exist even if the function being represented is only **piecewise** continuous. To identify the effect of a jump, it is convenient to introduce a **delta function** as part of the derivative of a function with a jump discontinuity. While the Fourier coefficients of any honest function must tend to 0 as $n \to \infty$, the coefficients of the delta function tend to be of a fixed size. This will allow the Fourier coefficients of functions having a piecewise definition with each piece given by a polynomial can be found without explicit calculation of any but the simplest integrals.

1. Real Fourier series

Given a function $f(x)$ on the interval $[-p, p]$, its **Fourier series** is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right)$$

where the **Fourier coefficients** a_n and b_n are defined by

$$\begin{aligned} a_0 &= \frac{1}{p} \int_{-p}^{p} f(x)\, dx \\ a_n &= \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx \\ b_n &= \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi x}{p}\, dx \end{aligned}$$

The **orthogonality** if the functions

$$1, \cos \frac{n\pi x}{p}, \sin \frac{n\pi x}{p}$$

assures us that the Fourier series of one of these functions is just the single term that is **the function itself**. In particular, the Fourier series of a constant function is itself.

Also note that the constant term of the series $a_0/2$ is the **average** value of $f(x)$.

Although it is customary to use only the **expression defining** $f(x)$ in describing exercises, the result depends **in an essential way** on the interval $[-p, p]$. The functions in the Fourier series for this interval are all **periodic** with period $2p$, so the actual function represented by the series is a **periodic extension** of $f x$ to all of $\mathbb{R}$. Different values of p usually lead to different extensions. Once the function is extended to be periodic, the integrals defining the Fourier coefficients can be computed using **any** interval of length $2p$. The use of a symmetric interval allows some saving in the case of **even functions** having only a **cosine series** or **odd functions** having only a **sine series**.

Another consequence of the need to use a periodic extension is that a function for which $f(p) \neq f(-p)$ must be considered as having a **jump discontinuity** at $x = p$.

2. Examples

Let

$$f(x) = \begin{cases} 1 & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

for some a with $0 < a < p$. This is an even function, so $b_n = 0$ for all n and

$$\begin{aligned} a_n &= \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx \\ &= \frac{1}{p} \int_{-a}^{a} \cos \frac{n\pi x}{p}\, dx \\ &= \frac{1}{p} \left[\frac{p}{n\pi} \sin \frac{n\pi x}{p} \right]_{-a}^{a} \\ &= \frac{2}{n\pi} \sin \frac{n\pi a}{p} \end{aligned}$$

for $n > 0$. The exclusion of $n = 0$ is an **essential** restriction since the expression for the integral has a factor of n in its denominator. A separate (easier) computation gives $a_0 = 2a/p$. Since this is a pure cosine series, the a_n computed here give the **Frequency Spectrum** defined in section 12.4 of the textbook. A special case of this example is Example 3 of that section, illustrated in Figure 12.19.

Let $g(x) = x$ on the interval $[-p, p]$. Except in the trivial case $p = 0$, we have $g(p) \neq g(-p)$, so this function must be considered as having a jump discontinuity at $x = p$. Note that the location of the discontinuity depends on p, so the common expression describes different function, with different series, for different p. This is an odd function, so all $a_n = 0$ and

$$\begin{aligned} b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx \\ &= \frac{1}{p}\int_{-p}^{p} x\sin\frac{n\pi x}{p}\,dx \\ &= \frac{1}{p}\left[\frac{p^2}{n^2\pi^2}\sin\frac{n\pi x}{p} - \frac{p}{n\pi}x\cos\frac{n\pi x}{p}\right]_{-p}^{p} \\ &= \frac{(-1)^{n+1}2p}{n\pi} \end{aligned}$$

Details have been omitted, but it is easy to see that our expression for the indefinite integral is correct (we will describe a method of computing it later). To get the definite integral, we use $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ for integer n.

Again, because this is a pure sine series, these coefficients give the Frequency Spectrum.

Note that both of these examples have Fourier coefficients that are a bounded quantity divided by n. Convergence of such series is often difficult to establish since such series are not **absolutely convergent**.

3. Complex Fourier series

Because the even and odd parts of functions often have significance in applications, it is sometimes convenient to have two types of term, sines and cosines, appear in the series. Unfortunately, this means that almost everything will require the discussion of several cases in proofs. The use of the identity

$$e^{ix} = \cos x + i\sin x$$

and its elementary consequences allows a unified approach to the theory at the expense of calculating with complex numbers. We will see that this only means computing a single complex quantity $c_n = (a_n - b_n i)/2$ in place of the two real numbers a_n and b_n, so the difference is mostly only a matter of bookkeeping. Although we will also require c_{-n} in order to describe the series, we have $c_{-n} = (a_n + b_n i)/2$, so the two complex numbers $c_{\pm n}$ are **complex conjugates** for every **real** function $f(x)$. The resulting series is

$$f(x) = \sum_{-infty}^{\infty} c_n e^{in\pi x/p}$$

It is the need for this result that justifies the division by 2 and choice of signs in the relation between the c_n and the coefficients a_n and b_n of the real series.

Combining our definitions,

$$\begin{aligned} c_n &= \frac{1}{2p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\,dx - \frac{i}{2p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx \\ &= \frac{1}{2p}\int_{-p}^{p} f(x)\left(\cos\frac{n\pi x}{p} - i\sin\frac{n\pi x}{p}\right)dx \\ &= \frac{1}{2p}\int_{-p}^{p} f(x)e^{-n\pi i x/p}\,dx \end{aligned}$$

Although $n > 0$ was assumed in this calculation, the result is valid for all n, since $\cos x$ is an even function and $\sin x$ is an odd function.

The computation of c_n using this expression applies to complex functions. If $f(x) = e^{m\pi i/p}$ for $m \neq n$, then

$$\begin{aligned} c_n &= \frac{1}{2p}\int_{-p}^{p} e^{m\pi i/p} e^{-n\pi i x/p}\,dx \\ &= \frac{1}{2p}\int_{-p}^{p} e^{(m-n)\pi i x/p}\,dx \\ &= \frac{1}{2p}\frac{p}{(m-n)\pi i}(e^{(m-n)\pi i} - e^{-(m-n)\pi i}) \\ &= 0 \end{aligned}$$

since the exponents differ by an integer multiple of 2π ($m \neq n$ was assumed when we wrote a formula requiring division by this quantity). When $m = n$, the integral reduces to the integral of the constant function 1 over an interval of length $2p$ divided by the length of the interval. so it is 1. Again, we see that each basic function, and hence each **linear combination** of basic functions, has a Fourier series that is itself.

Another example is the function $h(x) = e^x$ on $[-\pi, \pi]$. For this function

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{p} \pi e^x e^{-nix}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-ni)x}\,dx \\ &= \frac{1}{2\pi}\left[\frac{1}{1-ni}e^{(1-ni)x}\right]_{-\pi}^{\pi} \\ &= \frac{(-1)^n(e^{\pi} - e^{-\pi})}{2\pi(1-ni)} = \frac{(-1)^n(e^{\pi} - e^{-\pi})(1+ni)}{2\pi(1+n^2)} \end{aligned}$$

A direct computation of the a_n and b_n would be more complicated.

Since it is easy to convert between the real and complex Fourier series, only one needs to be computed in any example. Theoretical results are typically easier to describe using the complex series, but the real series is often easier to work with in explicit examples.

For real functions, c_n and c_{-n} are **complex conjugates**. When a_n and b_n are recovered from the complex series, we have $a_n = c_n + c_{-n}$, so it is twice the **real part** of c_n. Similarly, $b_n = i(c_n - c_{-n})$, making it twice the **imaginary part** of c_{-n}.

4. Scaling

Several techniques that are analogs of the operational properties of the Laplace transform require comparison of the Fourier coefficients of different functions. The notation we acquired from the textbook contains no mention of the function, so we must invent a notation that will allow us to state results. We propose to denote the complex Fourier coefficients of $f(x)$ by $[f]c_n$, with a similar convention for other functions, so that those of $g(x)$ will be $[g]c_n$.

Suppose we are given a function f on the interval $[-p, p]$, and use it to construct a function g using the formula $g(x) = f(bx)$. For the argument bx of the function f to satisfy $-p \leq bx \leq p$, we must have $-p/b \leq x \leq b/p$, so g needs to be considered on the interval $[-p/b, p/b]$. Then, on the appropriate intervals, using the substitution $y = bx$, $dy = b\,dx$,

$$\begin{aligned}[g]c_n &= \frac{b}{2p}\int_{-p/b}^{p/b} g(x)e^{-in\pi bx/p}\,dx \\ &= \frac{b}{2p}\int_{-p/b}^{p/b} f(bx)e^{-in\pi bx/p}\,dx \\ &= \frac{1}{2p}\int_{-p}^{p} f(y)e^{-in\pi y/p}\,dy \\ &= [f]c_n\end{aligned}$$

Note that the series for $g(x)$ is obtained by replacing x by bx in the series for $f(x)$.

5. A shifting theorem

Let functions of period $2p$ be related by $g(x) = f(x-b)$. Then, using P to denote any interval of length $2p$ and the substitution $u = x - b$,

$$\begin{aligned}[g]c_n &= \frac{1}{2p}\int_{x\in P} g(x)e^{-in\pi x/p}\,dx \\ &= \frac{1}{2p}\int_{x\in P} f(x-b)e^{-in\pi x/p}\,dx \\ &= \frac{1}{2p}\int_{u+b\in P} f(u)e^{-in\pi(u+b)/p}\,dx \\ &= \frac{e^{-n\pi ib/p}}{2p}\int_{u+b\in P} f(u)e^{-n\pi iu/p}\,dx \\ &= e^{-n\pi ib/p}[f]c_n\end{aligned}$$

A consequence of this is that f and g have the same Frequency Spectrum because the c_n differ by a **factor** of absolute value 1.

Note that the formula relating the c_n says that the **Fourier series** for $g(x)$ is found by writing $x - b$ in place of x in the Fourier series for $f(x)$, as one would expect from the notation that says that a function is **equal** to its Fourier series.

6. Delta functions

The operational view of a transform is enhanced by enlarging the realm of objects to which the transform may be applied. We have already seen how both the theory and applications of the Laplace transform are aided by the invention of $\delta(x)$, defined to be zero except at $x = 0$ and **just infinite enough** at zero that its integral over any interval including zero is 1.

Since Fourier coefficients are defined by integrals like those used to define the Laplace transform, we have the Fourier series

$$\delta(x) = \frac{1}{2p}\sum_{n=-\infty}^{\infty} e^{in\pi x/p}.$$

An impulse at $x = b$ is indicated by

$$\begin{aligned}\delta(x-b) &= \frac{1}{2p}\sum_{n=-\infty}^{\infty} e^{in\pi(x-b)/p} \\ &= \frac{1}{2p}\sum_{n=-\infty}^{\infty} e^{-inb\pi/p}e^{in\pi x/p}.\end{aligned}$$

In particular, when $b = p$, $2pc_n = e^{-in\pi} = (-1)^n$.

7. Differentiation

We consider a function f that is both **piecewise continuous** and **piecewise differentiable**; that is, the period interval can be written as a union of a finite number of closed intervals such that the function is differentiable on the interior of each of those intervals. Visually, this says that the graph of the function is smooth except for a finite number of **jumps** or **corners**. The Fourier coefficients of this function will be a sum of integrals over each of the intervals on which the function is nice. On each of those intervals $[a,b]$, consider the **integration by parts** formula obtained from the derivative of $f(x)e^{-n\pi ix/p}$:

$$\int_a^b f'(x)e^{-n\pi ix/p}\,dx = \Big[f(x)e^{-n\pi ix/p}\Big]_a^b + \frac{n\pi i}{p}\int_a^b f(x)e^{-n\pi ix/p}\,dx.$$

The complex Fourier coefficients of $f'(x)$ are found by adding the terms on the left over all intervals $[a,b]$ and dividing by $2p$.

The second terms on the right combine in the same way as the terms on the left to give the complex Fourier coefficients of $f(x)$ **multiplied by** $n\pi i/p$. This factor is exactly the factor expected from term-by-term differentiation of the Fourier series.

In the first term on the right, each interval contributes the limit of the value of $f(x)e^{-n\pi ix/p}$ at the point with a positive sign if it is a right endpoint and a negative sign if it is a left endpoint. When we consider the sum of **all** of these terms, there will be a contribution of

$$e^{-n\pi ia/p}\Big(\lim_{x\to a^-} f(x) - \lim_{x\to a^+} f(x)\Big)$$

at each endpoint a. This is $-e^{-n\pi ia/p}$ times the size of the jump at a (zero at points where f is continuous, even if it is a corner). If this term is moved to the other side of the equation, it can be identified with the product of $2p$ with the size of the jump and the Fourier coefficient of $\delta(x-a)$. This agrees with our interpretation of the delta function as the derivative of a function that is constant except for a jump of unit size. Thus, provided that the Fourier series of delta functions are introduced to represent the derivative of the jumps, the Fourier series of f' is the term by term derivaive of the Fourier series of f.

In our first example, $f(x)$ is constant except for a unit jump upward at $x = -a$ and downward at $x = a$. Thus, we may consider $f'(x) = \delta(x+a) - \delta(x-a)$. This is an odd function, so it gives a Fourier series

$$f'(x) = \frac{-2}{p}\sum_{n=1}^{\infty} \sin\frac{n\pi a}{p}\sin\frac{n\pi x}{p}$$

Integrating term-by-term gives the same result as we originally obtained for a_n when $n > 0$. The value of a_0 must be calculated separately as before.

In the second example, $g'(x) = 1 - 2p\delta(x-p)$. This is an even function, so it gives the Fourier series.

$$g'(x) = -2\sum_{n=1}^{\infty} \cos n\pi \cos\frac{n\pi x}{p}$$

The a_0 term is zero because the jump in $g(x)$ at the end of the interval exactly balances the growth over the rest of the interval. Indeed, the derivative of **any** periodic function must have constant term zero. Again, term-by-term integration gives the previous series for $g(x)$.

Since two functions that differ by a constant have the same derivative, it must be impossible to determine $[g]c_0$ from the Fourier series of $g'(x)$. A separate computation of the **average value** of g is required to find $[g]c_0$.

8. Exercises

A. Use linearity and the results of examples in Section 2 of this document to find the Fourier series of $f(x) = 5x + 3$ on the interval $[-\pi, \pi]$.

B. Consider the following three functions on $[-\pi, \pi]$, each of which takes the value $+1$ $1/6$ of the time, -1 $1/6$ of the time, and 0 the remaining $2/3$ of the time:

$$f(x) = \begin{cases} 1 & 0 < x < \pi/3 \\ -1 & -\pi/3 < x < 0 \\ 0 & \text{otherwise} \end{cases}, \qquad g(x) = \begin{cases} 1 & \pi/3 < x < 2\pi/3 \\ -1 & -2\pi/3 < x < -\pi/3 \\ 0 & \text{otherwise} \end{cases},$$

$$h(x) = \begin{cases} 1 & \pi/3 < x < 2\pi/3 \\ -1 & -\pi/3 < x < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Use linearity and the results of examples in this document to find the **complex Fourier series** and the **Frequency Spectrum** of each of these functions.

C. Consider the function on $[-1, 1]$ defined by

$$f(x) = \begin{cases} 0 & -1 < x \le 0 \\ x & 0 < x \le 1 \end{cases}.$$

This expression appears continuous on $[-1, 1]$, but its **periodic extension** of period 2 has a discontinuity at $x = 1$ since

$$\lim_{x \to 1^-} = 1 \quad \text{and} \quad \lim_{x \to -1^+} = 0$$

Modify the examples of Section 7 to find the Fourier series of f' including a multiple of $\delta(x-1)$ that must be the term by term derivative of the Fourier series of f, and use this to determine the series for $f(x)$ **except for its constant term**. Then, evaluate $c_0 = a_0/2$ as the **average value** of $f(x)$ on the interval $[-1, 1]$. Put these together to get the whole Fourier series for $f(x)$.

End of supplement