Mathematics 421 Essay 3
Boundary Value Problems
Spring 2008

0. Introduction

A **second order** differential equation has a general solution containing two parameters. Typically these parameters are the values of the solution and its first derivative at a single point. Under suitable conditions, the theory predicts that such data leads to a unique solution. However, some natural questions lead to the value of the function at **two different points** being specified. In such questions, the function restricted to **the interval between those points** is the main object of interest, so these questions are called **boundary value problems**. If the equation is linear and homogeneous, and the given boundary values are zero, then a unique solution could only be the zero function. However, **there is no uniqueness theorem for boundary value problems**. Indeed, certain homogeneous equations with zero boundary values have non-trivial solutions.

1. The main example

A typical example is

$$\frac{d^2y}{dx^2} + \lambda y = 0; y(0) = 0; y(L) = 0$$

for fixed L and a parameter λ. If $\lambda < 0$, write $\lambda = -\alpha^2$. Then, the general solution of the differential equation is $y = ae^{\alpha x} + be^{-\alpha x}$. The condition at $x = 0$ requires $b = -a$, so the solution is a multiple of $\sinh x$. This function is strictly increasing, so the condition at $x = L$ allows only the zero function. If $\lambda = 0$, the solution is $y = a + bx$, the condition at $x = 0$ gives $a = 0$, and again the solution is a multiple of the increasing function $y = x$, and only $b = 0$ allows $y(L) = 0$. If $\lambda > 0$, write $\lambda = \alpha^2$. Then, the general solution of the differential equation is $y = a\cos\alpha x + b\sin\alpha x$, and the condition at $x = 0$ gives $a = 0$. However, if $\alpha = n\pi/L$, giving $\lambda = n^2\pi^2/L^2$, all multiples of $\sin\alpha x$ satisfy the condition at $x = L$.

The functions that appear in these solutions are exactly the **odd functions** on $-L \le x \le L$ appearing in **half range Fourier series**.

2. Eigenfunctions

There is something more general than Fourier series involved here. The boundary condition at each endpoints is allowed to be the requirement that some fixed linear combination of y and y' be zero at that point. For example, the boundary value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0; y(0) = 0; y(L) + y'(L) = 0$$

has a non-trivial solution of the form $\sin\alpha x$ when $\sin\alpha L + \alpha\cos\alpha L = 0$. There are still infinitely many such α, but they are characterized by $\alpha = -\tan\alpha L$ — an equation that has one root in each interval of the form $\left[(k - {}^1\!/_2)\pi/L, (k + {}^1\!/_2)\pi/L\right]$. As before, $\lambda = \alpha^2$, but there is no simple expression for α. It can be shown by the method used in the first example that only these positive values of λ allow nonzero solutions of the boundary value problem.

In these problems, we are considering the effect of the **linear operator** d^2/dx^2 on the **linear space** of functions defined on the interval $[0, L]$ satisfying certain **homogeneous conditions** at the **boundary points** $x = 0$ and $x = L$. We identified functions taken into **multiples of themselves** by the operator. In **vector spaces**, such objects were called **eigenvectors** with the multiplier being called an **eigenvalue**, and in these **function spaces** the usual name is **eigenfunction**. In finite dimensional vector spaces, operators were characterized (in most cases) by their eigenvalues and eigenvectors, so that the behavior at a general vector could be found from the eigenvalues and eigenvectors. We aim for a corresponding result in function spaces.

Computers allow a systematic treatment of these more general problems, and you will use this resource in applications. However, homework and exams will emphasize problems whose eigenfunctions resemble Fourier series because they are more familiar and often lead to formulas for the solution.

The properties of the problem that allow a systematic study of its solution are that the **second derivative operator** is linear on the space of all smooth functions on $[0, L]$. The differential equation asserts that this operator takes y to a multiple of itself. In linear algebra, such multiples (which are λ for the negative of the second derivative operator) are called **eigenvalues** of the operator and the **nonzero** vectors multiplied by an eigenvalue are called **eigenvectors**. When considering operators on spaces of functions, we refer instead to **eigenfunctions**.

3. Self-adjoint operators

In a finite dimensional space, the eigenvalues and eigenvectors of an operator given by a **symmetric matrix** have special properties. All eigenvalues are **real numbers** and there is an **orthogonal basis** of eigenvectors. The analog for differential operators on a space of functions on an interval I is an equation in **self-adjoint form**

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) + (q(x) + \lambda p(x))y = 0$$

where $p(x)$ and $r(x)$ are **positive** on I. The presence of the function $p(x)$ gives a generalization of the usual eigenvalue problem, and the eigenfunctions will be orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_I p(x) f(x) g(x)\, dx.$$

The proof of Theorem 12.3(d) in the text shows how one works with this more general setting.

4. Integrating factors

Although the self-adjoint form looks special, an arbitrary second order operator can be put in that form simply by multiplying by a suitable function that allows the terms of degree one and two to have the required form. As in other examples of this methods, this function is called an **integrating factor**. Indeed, the method of discovery and the expression for this integrating factor are similar to case of the first order linear equation. Multiplying $y'' + b(x)y'$ by $r(x)$ gives the derivative of $r(x)y'$ precisely when $r' = br$, so $b(x)$ must be the derivative of $\ln r(x)$. Thus r is given by integrating $b(x)$ and exponentiating the result. On any interval where this can be done, e.g., an interval where $b(x)$ is continuous, the result is **positive**, as required by the general theory.

5. The Sturm-Liouville Theory

The orthogonality of eigenfunctions with respect to the positive weight function $p(x)$ proved in Theorem 12.3(d) allows endpoints where $r(x) = 0$ as well as those where a boundary condition is prescribed. This allows examples for which the equation is **singular** at one or both boundary points and the solution is required to satisfy a condition at that point limiting attention to a one-dimensional space of functions. The text gives examples arising from Bessel's equation and Legendre's equation. The nature of the boundary condition leads to simplicity of eigenvalues noted in Theorem 12.3(b). The linear independence of eigenfunctions for different eigenvalues claimed by Theorem 12.3(c) is a general property that has an easy proof based of a trick: one supposes that a **simplest** dependence relation has been found; the operator is applied to it and the result simplified using the fact that the terms are eigenfunctions; these two dependence relations are then combined to get a simpler non-trivial relation. A deeper study is required for part (a) of the theorem.

Such a study can be found in Hans Sagan, “Boundary and Eigenvalue Problems in Mathematical Physics”, Dover Publications, NY (my copy has a price of $17.95 on the cover — Dover aims to publish inexpensive paperback books, many of which were textbooks abandoned by their original publisher). Although a general proof may be difficult, the verification of the properties is easy in any particular case.

It is conventional to let the eigenvalue be the λ in the equation, so the operator is the **negative** of the sum of the other terms, i.e.,

$$-\left(\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right)+q(x)y\right)$$

This hides the quantity $p(x)$, but this is the weight function, so it will appear in the orthogonality relation. Indeed, this weight function should **not** be hidden since it identifies the problem as an extended form of the eigenvalue problem that is considered for symmetric matrices.

6. The parametric Bessel equation

The Bessel function of the first kind of order $\nu \geq 0$, denoted $J_\nu(x)$, satisfies

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\nu^2)y=0$$

with

$$\lim_{x\to 0} x^{-\nu}J_\nu(x)$$

equal to a particular finite, nonzero value. The **method of Frobenius** produces a **series solution** with this property. See sections 5.6 through 5.8 of the Math 244 text by Boyce and DiPrima or sections 5.2 and 5.3 of the present textbook for details.

The singular point at zero is a natural location for one of the boundaries. To select another boundary, we note that $J_\nu(x)$ is an oscillatory function, so setting λ equal to the successive solutions of $J_\nu(x)=0$ allow $x=1$ to be a boundary at which $J_\nu(\lambda x)=0$. Thus, the functions $J_\nu(\lambda x)=0$ with fixed ν and boundary conditions requiring $\lim_{x\to 0} x^{-\nu}f(x)$ to be finite and $f(1)=0$ leads to this sequence of functions. It remains to find the differential equation satisfied by the $J_\nu(\lambda x)$, to use that equation to characterize these as eigenfunctions of some operator, and to show that these functions form a **complete** basis for functions satisfying the boundary conditions.

We find an equation satisfied by this sequence of functions and put it in **self-adjoint form**, but omit the proof of completeness..

Writing $u=\lambda x$ and $w=J_\nu(u)$, we have

$$\frac{du}{dx}=\lambda \quad \text{and} \quad u^2\frac{d^2w}{du^2}+u\frac{dw}{du}+(u^2-\nu^2)w=0$$

Now,

$$\frac{dw}{dx}=\frac{dw}{du}\frac{du}{dx}=\lambda\frac{dw}{du}$$

and

$$x\frac{dw}{dx}=\lambda x\frac{dw}{du}=u\frac{dw}{du}.$$

Similarly,

$$x^2\frac{d^2w}{dx^2}=u^2\frac{d^2w}{du^2}.$$

Thus,

$$x^2\frac{d^2w}{dx^2}+x\frac{dw}{dx}+(\lambda^2x^2-\nu^2)w=0$$

This equation is known as **the parametric Bessel equation**.

The integrating factor to put this in self-adjoint form is $1/x$, so that form of the equation is

$$\frac{d}{dx}\left(x\frac{dw}{dx}\right) + (\lambda^2 x - \frac{\nu^2}{x})w = 0,$$

and the eigenfunctions will be orthogonal on $[0, 1]$ with respect to the weight x.

7. Legendre polynomials

The Legendre polynomials satisfy an equation whose self-adjoint form is

$$\frac{d}{dx}\left((1 - x^2)\frac{dy}{dx}\right) + n(n + 1)y = 0$$

General properties of series solutions about $x = 0$ show that one solution is a polynomial of degree n. Furthermore $x = \pm 1$ are singular points of the equation, and the **method of Frobenius** shows that these polynomials are the only solutions bounded a these singular points. Using the Legendre polynomials $P_n(x)$ as eigenfunctions with eigenvalue $n(n + 1)$ gives a family of orthogonal functions on $[-1, 1]$.

8. How will this be used?

Eigenfunction expansions generalize Fourier series and are determined using similar methods. Indeed, some are variants on Fourier series corresponding to particular boundary conditions. However, this is not reason enough to introduce them.

The power of eigenfunction expansions lies in their ability to represent all reasonable functions satisfying certain boundary conditions and to **have a predictable behavior when differentiated**. This will justify the **method of separation of variables** for solving certain **partial differential equations**. The theory of partial differential equations emphasizes boundary value problems since there are only a limited class of physical situations allowing anything like the initial value problems that were used to organize the study of ordinary differential equations. A good beginning to this study is a consideration of the role of boundary data in ordinary differential equations.

Initially, we will consider problems on **rectangular regions in the plane**, and we will give complete solutions to some classical equations in that context. In order to apply our methods to other regions, the problems will need to be described in a way that is independent of coordinates, and then introduce a special coordinate system to describe the region on which we want to solve the equation. The parametric Bessel functions and Legendre polynomials will play a role in the study of important regions in two and three dimensions.