Mathematics 421 Essay 1

Using the Laplace transform

Spring 2006

0. Introduction

The **Laplace transform** of a function of t is a function of a new variable s defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

In section 4.1 of the textbook, an example was given in which this definition was easy to use:

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}.$$

The special case with $a = 0$ should be noted. However, it will turn out that even these examples are consequences of general properties of the transform, so that the definition will **only** be used to derive general properties. In stating these properties, only the simplest version will be shown; repeated application will be done as needed instead of used to state results with proofs requiring **mathematical induction**. Since the main applications involve **very few steps**, nothing is gained in most cases by pretending that there is a general formula.

1. Linearity

The most important property of the Laplace transform is **linearity**. This is a direct consequence of the linearity of integration. The basic statements are

$$\begin{aligned} \mathcal{L}\{f(t)\} = F(s) &\implies \mathcal{L}\{c\,f(t)\} = c\,F(s) \\ \mathcal{L}\{f(t)\} = F(s) \text{ and } \mathcal{L}\{g(t)\} = G(s) &\implies \mathcal{L}\{f(t) + g(t)\} = F(s) + G(s) \end{aligned}$$

Repeated use of this rule deals with a sum of **arbitrarily many** terms, each of which is a product of a constant and a known function. The generalization to such expressions has been common since the first course in algebra. Such general expressions are called **linear combinations** of the known function.

In addition to determining transforms, it will be necessary to find **inverse** transforms. Thus, any function that can be written as a linear combination of $1/(s-a)$ can be recognized as the Laplace transform of a linear combination of e^{at}. The method of **partial fractions** produces such an expression from **some** quotients of polynomials. Quotients of polynomials are called **rational functions**; and a rational function is called **proper** if the degree of the numerator is **strictly smaller** than the degree of the denominator. The functions that are Laplace transforms of linear combinations of exponentials are **proper** rational functions whose denominator is a product of distinct factors of the form $x - a$.

In the first course on Differential Equations, solutions of linear differential equations with constant coefficients were found by **assuming a solution** of the form $y = e^{at}$. Some equations had solutions that were trigonometric functions, and these could be found using Euler's identity $e^{it} = \cos t + i \sin t$. This leads to

$$\cos t = \frac{e^{it} + e^{-it}}{2} \quad \text{and} \quad \sin t = \frac{e^{it} - e^{-it}}{2i}$$

If we accept these formulas, then

$$\mathcal{L}\{\cos t\} = \frac{1}{2}\left(\frac{1}{s-i} + \frac{1}{s+i}\right) = \frac{s}{s^2+1}$$
$$\mathcal{L}\{\sin t\} = \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) = \frac{1}{s^2+1}$$

These formulas can also be obtained by characterizing the trigonometric functions as solutions of **initial value problems**. We illustrate this in section 3.

2. Derivatives

If you use **integration by parts** in the definition of $\mathcal{L}\left\{f'(t)\right\}$, you get

$$\int_0^\infty f'(t)e^{-st}\,dt = e^{-st}f(t)\big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt = -f(0) + s\mathcal{L}\{f(t)\}$$

provided that $\lim_{t\to\infty} f(t)e^{-st} = 0$ for sufficiently large s (such $f(t)$ are said to be **of exponent order**, and is a necessary condition for the existence of the Laplace transform).

Theoretical aside.The existence of the Laplace transform of a function of exponential order is Theorem 4.2 of the textbook.Theorem 4.5 refines this method of proof to show that **the transform always approaches zero** as $s \to \infty$. In the case in which the transform is a rational function, this says that it is always a **proper** rational function. The partial fraction decomposition will then be a sum of **proper** partial fractions.

Repeated use of this formula gives expressions for the Laplace transform of derivatives of any order, but it is probably easier to invoke this formula twice to get an expression for $\mathcal{L}\left\{f''(t)\right\}$ than to remember the resulting formula.

An important application of this is the use of Laplace transforms to solve **initial value problems**. Suppose that $y(t)$ satisfies a linear differential equation with constant coefficients whose right side has a known Laplace transform, together with initial conditions that serve to define $y(t)$ uniquely. Assume that $\mathcal{L}\{f(t)\} = Y(s)$. Then, the Laplace transform of the left side of the equation is the product of a polynomial in s times $Y(s)$ plus another polynomial in s. Equating this to the Laplace transform of the right side gives a linear **algebraic** equation for $Y(s)$. If the Laplace transform of the right side is a rational function, then $Y(s)$ will also be a rational function. The solution of the initial value problem is reduced to **partial fractions** and some basic examples of Laplace transforms.

Linearity tells us that $\mathcal{L}\{0\} = 0$, so the rule for derivatives gives

$$0 = \mathcal{L}\{0\} = s\mathcal{L}\{1\} - 1$$

from which we conclude the previous result that $\mathcal{L}\{1\} = 1/s$. The transform of higher powers can be found by **mathematical induction** using the derivative formula for $f(t) = t^n$ which is

$$n\mathcal{L}\left\{t^{n-1}\right\} = s\mathcal{L}\left\{t^n\right\}$$

for $n > 0$.

Applying the rule that Laplace transforms tend to zero as $s \to \infty$ to $\mathcal{L}\left\{f'(t)\right\}$ tells us that

$$\lim_{s\to\infty} sF(s) = f(0)$$

This is an example of information about a function being **visible** in its Laplace transform.

3. Examples

The function e^{at} is the solution $y(t)$ of the **initial value problem**

$$\frac{dy}{dt} - ay = 0, \qquad y(0) = 1$$

If $\mathcal{L}\{y(t)\} = Y(s)$, then the initial condition gives $\mathcal{L}\left\{y'(t)\right\} = sY(s) - 1$, and the equation asserts that $(s-a)Y(s) - 1 = 0$. Thus, $\mathcal{L}\left\{e^{at}\right\} = 1/(s-a)$, as has already been noted.

Similarly, $\cos t$ is the solution $y(t)$ of the **initial value problem**

$$\frac{d^2y}{dt^2} + y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

If $\mathcal{L}\{y(t)\} = Y(s)$, then the initial conditions give $\mathcal{L}\left\{y'(t)\right\} = sY(s) - 1$ and $\mathcal{L}\left\{y''(t)\right\} = s(sY(s) - 1) = s^2Y(s) - s$, and the equation asserts that $(s^2+1)Y(s) - s = 0$.

Another simple example is $\sin t$, which is the solution $y(t)$ of the **initial value problem**

$$\frac{d^2y}{dt^2} + y = 0, \qquad y(0) = 0, \qquad y'(0) = 1$$

If $\mathcal{L}\{y(t)\} = Y(s)$, then the initial conditions give $\mathcal{L}\left\{y'(t)\right\} = sY(s)$ and $\mathcal{L}\left\{y''(t)\right\} = s(sY(s)) - 1 = s^2Y(s) - 1$, and the equation asserts that $(s^2+1)Y(s) - 1 = 0$.

4. Scaling the argument

Consider functions $f(t)$ and $g(t)$ related by $g(t) = f(bt)$ for some constant b. Denote the transform of $f(t)$ by $F(s)$. Then, the substitution $u = bt$ gives

$$\begin{aligned}
G(s) = \mathcal{L}\{g(t)\} &= \int_0^\infty g(t)e^{-st}\,dt \\
&= \int_0^\infty f(bt)e^{-st}\,dt \\
&= \int_0^\infty f(u)e^{-su/b}\,\frac{1}{b}du \\
&= \frac{1}{b}\int_0^\infty f(u)e^{-(s/b)u}\,du \\
&= \frac{1}{b}F\big((s/b)\big)
\end{aligned}$$

As in this computation, the convention of using the corresponding upper case letter to name the Laplace transform of a function named by a lower case letter is used throughout this subject. Here are some examples of scaling:

$$\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \frac{1}{a}\frac{1}{(s/a)-1} = \frac{1}{s-a} \\
\mathcal{L}\{\cos kt\} &= \frac{1}{k}\frac{s/k}{(s/k)^2+1} = \frac{1}{s^2+k^2} \\
\mathcal{L}\{\sin kt\} &= \frac{1}{k}\frac{1}{(s/k)^2+1} = \frac{k}{s^2+k^2}
\end{aligned}$$

This result also tells us the form of the transform of $f(t) = t^n$. If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\frac{1}{c^n} F(s) = \mathcal{L}\left\{\left(\frac{t}{c}\right)^n\right\} = cF(cs).$$

That is, multiplying s by c multiplies $F(s)$ by c^{-n-1}, so that $F(s)$ is of degree $-(n+1)$. Indeed, making the substitution $u = st$ in the integral defining $\mathcal{L}\{t^n\}$ gives the form of the transform and an expression for the constant factor for **arbitrary real** values of $n > 0$. Exercise 39 in section 4.2 asks for the details.

5. Multiplying by an exponential

Suppose $g(t) = e^{at} f(t)$. Then

$$G(s) = \int_0^\infty e^{-st} e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a)$$

In particular,

$$\mathcal{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2 + b^2} \quad \text{and} \quad \mathcal{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2 + b^2}$$

Expand these formulas to get simplify the expressions for the transforms. Use **completing the square** to return to this form when computing an **inverse Laplace transform**. In particular, combining this section with the previous one allows the Laplace transforms of all $t^n e^{at}\cos bt$ and $t^n e^{at}\sin bt$ to be found from the special cases in which $a = 0$ (and $b = 1$ if the trigonometric factor is present). Finding inverse transforms uses standard algebraic techniques to recognize the relation between the given expression and the transform of a simpler one.

6. Series

Assuming that the operations (essentially an interchange of limits) can be justified, the formula $\mathcal{L}\{t^n\} = n!s^{-n-1}$ leads to

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{a_n t^n}{n!}\right\} = \sum_{n=1}^{\infty} \frac{a_{n-1}}{s^n}$$

This allows **all** coefficients of the Taylor series for $f(t)$ about $t = 0$ to be related to those in a series expansion of $F(s)$ in powers of s^{-1} — which could be called a series expansion at infinity. The expansion of $f(t)$ is required to be **very strongly** convergent because of the $n!$ in the denominator, in order to allow the series for $F(s)$ to converge **anywhere**. However, these conditions are satisfied for the solutions of linear differential equations with constant coefficients that are the main examples used for $f(t)$. Laplace transforms are typically defined only for $s > c$ for some c, so a series for the transform should have the same property. This requires that the coefficients a_n should grow no faster than c^n.

Typical transform pairs are exponential functions in t, whose coefficients look like $c^n/n!$, and rational functions in s, whose coefficients behave like c^n.

7. Partial fractions

If a Laplace transform is a rational function has a denominator $\prod(s - \alpha_i)$, with n factors, then the numerator has degree less than n because the function must have limit zero as $s \to \infty$. Such rational functions are called **proper** by analogy to arithmetic proper fractions that have numerators that are smaller than their denominators. The inverse Laplace transforms of such expressions can be found using the **partial fraction decomposition** that was used to integrate such expressions. In the case in which all factors of the denominator are **different linear factors** the decomposition is easily found by the method that the text calls the "cover up method" (page 202). In this method, a proper rational function is written as a sum of simpler expressions **of the same type**:

$$\frac{P(s)}{(s-a)Q(s)} = \frac{A}{(s-a)} + \frac{P_1(s)}{Q(s)} \tag{$*$}$$

where $Q(a) \neq 0$ (indicating that $(s-a)$ does not divide $Q(s)$, and all fractions are proper fractions. Multiplying by $(s-a)$ and evaluating at $s=a$ gives

$$A = \frac{P(a)}{Q(a)}. \tag{$**$}$$

This works because a proper fraction with a linear denominator has a constant numerator.

The partial fraction decomposition is a special case of the fact that, if $Q_0(x)$ and $Q_1(x)$ are polynomials over a field (an algebraic system like the real number, complex numbers or rational numbers that allows addition, multiplication and division — except that division by zero is not allowed) have no common factors (other than constants), then there are polynomials A_0 and A_1 (with coefficients in the same field) such that

$$A_1Q_0 + A_0Q_1 = 1.$$

The proof uses the **Euclidean algorithm**, which also gives an efficient computation A_0 and A_1. If the degree of A_0 in such an equation is of the same degree as Q_0 or larger, then one can subtract a certain multiple of Q_0 from A_0 and add the same multiple of Q_1 to A_1 to get other choices of A_0 and A_1 with the same property and having the degree of A_0 less than the degree of Q_0. In this case, the degree of A_1 will automatically be less than the degree of Q_1. If this equation is divided by Q_0Q_1, the result is a partial fraction decomposition of $1/(Q_0Q_1)$ as a sum of proper fractions whose denominators are Q_0 and Q_1.

This equation also gives

$$(PA_1)Q_0 + (PA_0)Q_1 = P.$$

for any polynomial P. If the degree of P is less than the degree of Q_0Q_1, the use of division to replace PA_0 by a polynomial of degree less than the degree of Q_0, with a corresponding change in the other term, leads to a partial fraction decomposition of any proper fraction with denominator Q_0Q_1 as a sum of a proper fraction of denominator Q_0 and a proper fraction with denominator Q_1. All that is needed is that Q_0 and Q_1 have no common factor. In the simplest case of the Euclidean Algorithm, one gets

$$1 \cdot (x-a) + (-1) \cdot (x-b) = b - a.$$

The general method and the cover-up method are almost identical in this case.

Note that each linear factor of the denominator is determined separately by this method. If there are any irreducible quadratic factors or repeated linear factors in the denominator, they can be left until all simple linear factors have been removed by using $(**)$ to identify the numerator in the first term on the right side of $(*)$ for each simple linear factor and subtracting that term to leave a simpler fraction. In finding the numerators for each factor of the denominator, the original numerator $P(x)$ may be used for all factors with an appropriate choice of $Q(x)$ for each factor.

Linear fractions of higher multiplicity can be handled by the following variant on $(*)$

$$\frac{P(s)}{(s-a)^kQ(s)} = \frac{A}{(s-a)^k} + \frac{P_1(s)}{(s-a)^{k-1}Q(s)}$$

Here, multiplication by $(s-a)^k$ and evaluating at $s=a$ again gives $(**)$ although a factor of $(s-a)^{k-1}$ remains in the denominator. This allows a linear factor of multiplicity k to be removed in k steps, provided that the complementary term is found as part of each step.

If there is only one quadratic factor, the terms belonging to the other factors of the denominator can be found are subtracted from the original expression. When common factors are removed, the result will have only this factor in the denominator. However, there seems to be no easy way to deal with more than one quadratic factor.

8. An example

Exercise 10 in section 4.6 asks to use Laplace transforms to solve

$$\frac{dx}{dt} - 4x + \frac{d^3y}{dt^3} = 6\sin t$$
$$\frac{dx}{dt} + 2x - 2\frac{d^3y}{dt^3} = 0$$

with initial conditions $x(0) = y(0) = y'(0) = y''(0) = 0$. If $\mathcal{L}\{x(t)\} = X(s)$ and $\mathcal{L}\{y(t)\} = Y(s)$, then the initial conditions give $\mathcal{L}\big\{x'(t)\big\} = sX(s)$ and $\mathcal{L}\big\{y'''(t)\big\} = s^3Y(s)$, so Laplace transform of the equations may be written in the simple matrix form

$$\begin{bmatrix} s-4 & s^3 \\ s+2 & -2s^3 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{6}{s^2+1} \\ 0 \end{bmatrix}$$

The determinant of the coefficient matrix is

$$s^3\big(-2(s-4) - (s+2)\big) = -3s^3(s-2)$$

and the cofactor expression for the inverse gives

$$\begin{aligned}\begin{bmatrix} X \\ Y \end{bmatrix} &= \frac{-1}{3s^3(s-2)}\begin{bmatrix} -2s^3 & -s^3 \\ -s-2 & s-4-2s^3 \end{bmatrix}\begin{bmatrix} \frac{6}{s^2+1} \\ 0 \end{bmatrix} \\ &= \frac{2}{s^3(s-2)(s^2+1)}\begin{bmatrix} 2s^3 \\ s+2 \end{bmatrix}\end{aligned}$$

Thus,

$$X = \frac{4}{(s-2)(s^2+1)} = \frac{4/5}{s-2} - \frac{4/5\,(s+2)}{s^2+1}$$

where the first term is found by the cover-up method, and the second by simplifying the difference of the left side and the first term, which is

$$\frac{20-4(s^2+1)}{5(s-2)(s^2+1)} = \frac{16-4s^2}{5(s-2)(s^2+1)} = \frac{-8+4s}{5(s^2+1)}$$

From the basic transform pairs, we get

$$\begin{aligned} x(t) &= \frac{4}{5}e^{2t} - \frac{4}{5}\cos t - \frac{8}{5}\sin t \\ x'(t) &= \frac{8}{5}e^{2t} - \frac{8}{5}\cos t + \frac{4}{5}\sin t \\ y'''(t) &= \frac{8}{5}e^{2t} - \frac{8}{5}\cos t - \frac{6}{5}\sin t \end{aligned}$$

where the first line is the inverse transform of $X(s)$, the second line is found by differentiating $x(t)$ and the third line is the common value found by solving each equation in the original system algebraically for $y'''(t)$. This checks that $x(t)$ is a solution of the differential equations, and it is easy to see that it satisfies $x(0) = 0$. From this, one could integrate — keeping track of the initial conditions — to find $y(t)$. However, we want to illustrate the use of partial fractions. The cover up method produces terms of

$$\begin{aligned} \frac{1}{5(s-2)} &= \frac{s^3(s^2+1)}{5s^3(s-2)(s^2+1)} \\ \frac{-2}{s^3} &= \frac{-10(s-2)(s^2+1)}{5s^3(s-2)(s^2+1)} \end{aligned}$$

With the denominator of the expressions on the right, the original numerator of $Y(s)$, found in the matrix solution, is $10(s+2)$. Subtracting the two numerators above from this gives $-s^5 + 9s^3 - 20s^2 + 20s$. This is clearly divisible by s, and it must be divisible by $s - 2$ if our work is correct. Division reveals it to be $s(s-2)(-s^3 + 2s^2 + 5s - 10)$. The unidentified partial fractions add to

$$\frac{-s^3 - 2s^2 + 5s - 10}{5s^2(s^2+1)}$$

Another application of the cover up method gives a term of $-2/s^2 = -10(s^2+1) \;/\; \big(5s^2(s^2+1)\big)$. Subtracting this leaves $(-s^2 + 8s + 5) \;/\; \big(5s(s^2+1)\big)$. After one more step, we have the full partial fraction decomposition

$$Y(s) = \frac{1}{5}\frac{1}{s-2} - \frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s} + \frac{8-6s}{5(s^2+1)}$$

Taking inverse transforms gives

$$y(t) = \frac{1}{5}e^{2t} + 1 - 2t - t^2 + \frac{8}{5}\sin t - \frac{6}{5}\cos t.$$

From this, it is easy to obtain values of y', y'', and y''' to check all initial conditions and the previously discovered value of y'''.

9. The transform of a derivative

Assuming the validity of differentiating with respect to the parameter s under the integral sign, one has

$$F'(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty -tf(t)e^{-st}\,dt = -\mathcal{L}\{tf(t)\}.$$

In particular,

$$\begin{aligned} \mathcal{L}\{t \sin t\} &= -\frac{d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2} \\ \mathcal{L}\{t \cos t\} &= -\frac{d}{ds}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2} \end{aligned}$$

It should be **these** expressions with denominator $(s^2+1)^2$ that we should seek to obtain in a partial fraction expansion. Without a **simple mechanical method to produce such terms easily**, we cannot claim that Laplace transforms provide a superior method in cases where rational functions arise that have repeated quadratic factors in the denominator. Symbolic calculation systems like Maple have routines for handling this case, but there may not be a suitable general method for hand computation.

Another approach that may be suitable is to factor the denominator of a rational function over the complex numbers, and apply the previous methods for finding the coefficients. Although algebra with complex numbers is often much more cumbersome than you expect, the cover up method is well behaved in cases in which you know exact expressions for the roots of the denominator. In any case, expressions that are suitable for hand computation are of low enough degree that here are few opportunities for complicated expressions to arise.

10. Functions defined by cases

There is one more formula for Laplace transforms that is part of the general toolbox. If a function of t is zero for $t < a$ and given by some formula for larger t, the Laplace transform integral is best evaluated by the substitution $t = a + u$. This introduces a factor of e^{-as} and changes the integral to an integral in u from zero to infinity. This integral is the ordinary Laplace transform of the expression for our function in terms of u. To write this as a formula, we introduce the **Heaviside function** $\mathcal{U}(t-a)$, defined by

$$\mathcal{U}(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

Then, for $a \geq 0$,

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s).$$

Note that there is an implicit factor of $\mathcal{U}(t)$ whenever we are taking a Laplace transform since only values of $t > 0$ are considered when evaluating the integral in the definition of the Laplace transform.

Any function defined by cases may be written in an equivalent form using Heaviside functions. If the expression defining the function changes at $t = a$, a term is introduced with factor of $\mathcal{U}(t-a)$ multiplying the change from the expression used for $x < a$ to the one used for $x > a$.

To retrieve a definition by cases from one using Heaviside functions, the value in each interval is the sum of the expressions multiplying $\mathcal{U}(t-a)$ with smaller values of a.

When using this formula to find Laplace transforms, one must be careful to express the quantity multiplying $\mathcal{U}(t-a)$ as a function of $t-a$.

When finding inverse transforms, terms with the same e^{-as} factor are collected together and the inverse transforms of each of these clusters is found separately. In applications to differential equations, this usually finds a **continuous solution** even when there is a **discontinuous driving force**.

A convenient way to work with this is to use a graphical description of the function. If the graph of $f(t)$ if known, then the graph of $f(t-a)\mathcal{U}(t-a)$ is found by translating the known graph a units to the right.

As an example, consider

$$f(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ 2-t & \text{if } 1 < t < 2 \\ 0 & \text{if } t > 2 \end{cases}$$

whose graph consists of line segments from $(0,0)$ to $(1,1)$, from $(1,1)$ to $(2,0)$ and then a **ray** along the horizontal axis to the right.

Instead of working with formulas, we use a graphical method of finding $F(s)$ starting from $\mathcal{L}\{t\} = 1/s^2$. Translating this graph one unit to the right gives a parallel line having a Laplace transform of e^{-s}/s^2. Subtracting the second function from the first gives a line from $(0,0)$ to $(1,1)$ followed by a horizontal ray. The Laplace transform of this function is $\left(1-e^{-s}\right) \big/ s^2$. Translating **this graph** one unit to the right gives a line from $(1,0)$ to $(2,1)$ followed by a horizontal ray. The Laplace transform of this function is

$e^{-s}\left(1-e^{-s}\right) / s^2$. Subtracting the second function from the first gives the desired function and shows that its Laplace transform is $\left(1-e^{-s}\right)^2 / s^2$.

An extension of this method invents **generalized functions** like the **Dirac delta function** $\delta(t-a)$ that acts like a **derivative of the Heaviside function**. Physically, it plays the role of an **impulse** that effects an abrupt change in momentum. Its Laplace transform is e^{-as}. It may be used formally in a solution of differential equations by Laplace transforms and gives rise to a continuous solution of the equation. When made rigorous, this shows that any physically realizable force approximating an impulse leads to motion approximating this solution.

11. Convolution

There is one more operation inspired by the study of Laplace transforms. The **convolution** of $f(t)$ and $g(t)$, denoted by $(f * g)(t)$ is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

A change of variables in the sector defined by $0 < \tau < t < \infty$ in the (τ, t) plane shows that $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$. This is the subject of theorem 4.9 of the text. An application to the **Volterra integral equation** in Section 4.4 gives one application of the idea of convolution.

Little more needs to be added to the treatment in the textbook except to note that it may be easier the find the inverse transform of a product of two transforms than to compute a convolution directly.