Mathematics 373 Workshop 8 Solutions

Taylor methods

Fall 2003

Problem 1.

Consider the initial value problem

$$\frac{dy}{dt} = 2t - \frac{5t^2}{y} \qquad y(0) = 1. \tag{1}$$

The existence and uniqueness theorems break down when $y = 0$, so we will confine attention to the **window**

$$-2 \le t \le 2 \qquad 0 < y \le 5.$$

The solutions computed here remain inside this window, but an extended computation would be stopped when it left the window — through **any** edge.

1a Statement.

Verify that

$$\frac{d^2y}{dt^2} = -\frac{-2y^3 + 10ty^2 - 10t^3y + 25t^4}{y^3}$$

(This result was found using Maple).

1a Solution.

Differentiating without simplification gives

$$\frac{d^2y}{dt^2} = 2 - \frac{10t}{y} + \frac{5t^2\frac{dt}{dt}}{y^2}$$

Then, substituting the given expression for dy/dt gives

$$\frac{d^2y}{dt^2} = 2 - \frac{10t}{y} + \frac{5t^2\left(2t - \frac{5t^2}{y}\right)}{y^2}.$$

To clear fractions, multiply by y^3 to get

$$\begin{aligned} y^3\frac{d^2y}{dt^2} &= 2y^3 - 10ty^2 + 5t^2\left(2ty - 5t^2\right) \\ &= 2y^3 - 10ty^2 + 10t^3y - 25t^4 \end{aligned}$$

This is equivalent to the given expression,

1b Statement.

Use the information about dy/dt and d^2y/dt^2 from the equation and part (a) to show that every solution of the equation has a relative minimum point on the line $t = 0$ and a relative maximum point on the line $2y = 5t$.

1b Solution. The equation tells us that

$$\frac{dy}{dt} = 2t - \frac{5t^2}{y} = \frac{2ty - 5t^2}{y} = \frac{t(2y - 5t)}{y}$$

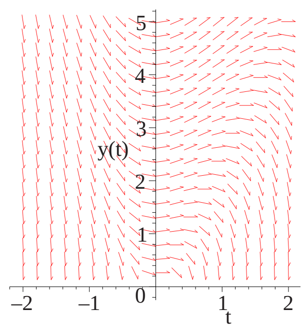
This is zero where $t = 0$ and where $2y = 5t$, so these lines pass through the **critical points** of the solutions $y(t)$. To determine whether the critical points are **relative minima** or **relative maxima**, use the **second derivative test**. If $t = 0$,

$$y^3 \frac{d^2y}{dt^2} = 2y^3,$$

so $d^2y/dt^2 = 2 > 0$, indicating a minimum. If $2y = 5t$,

$$\begin{aligned}\frac{d^2y}{dt^2} &= 2 - 10\frac{t}{y} + 10t\left(\frac{t}{y}\right)^2 - 25t\left(\frac{t}{y}\right)^3 \\ &= 2 - 10\frac{2}{5} + 10t\left(\frac{2}{5}\right)^2 - 25t\left(\frac{2}{5}\right)^3 \\ &= 2 - 4 + \frac{8t}{5} - \frac{8t}{5} = -2\end{aligned}$$

Here is a picture of the slope field for this equation in the given window.



The results calculated here are quite visible. Also note that the slope is vertical on the t axis.

1c Statement. Use **five steps** of a second order Taylor method starting from $(0, 1)$ with $h = 0.1$ to approximate the solution of the initial value problem (1). To check these values, use four steps of the same method starting from $(0, 1)$ with $h = 0.05$ and compare results.

1c Solution. Here are the values found by Maple.

0.	1.	0.	1.
0.1	1.010000000	0.05	1.002500000
0.2	1.030135892	0.10	1.008754478
0.3	1.051207504	0.15	1.017546262
0.4	1.064480339	0.20	1.027701319
0.5	1.060708853		

So far, the difference between the values of $y(t)$ is only about 0.0025. This is consistent with what would be expected from a second order method with $h = 0.1$ over an interval of length 0.2.

Problem 2. Using the corresponding fourth order method on (1) with $h = -0.01$ gives the following values:

0.	1.
−0.01	1.000101667
−0.02	1.000413330
−0.03	1.000944975
−0.04	1.001706559
−0.05	1.002708001
−0.06	1.003959161
−0.07	1.005469831
−0.08	1.007249713
−0.09	1.009308405
−0.10	1.011655379

2a Statement. Use the differential equation (1) to find the value of dy/dt at these points.

2a Solution. Here is the previous table augmented with a third column giving dy/dt.

0.	1.	0.
−0.01	1.000101667	−0.02049994917
−0.02	1.000413330	−0.04199917368
−0.03	1.000944975	−0.06449575163
−0.04	1.001706559	−0.08798637078
−0.05	1.002708001	−0.1124662414
−0.06	1.003959161	−0.1379290161
−0.07	1.005469831	−0.1643667182
−0.08	1.007249713	−0.1917696789
−0.09	1.009308405	−0.2201264864
−0.10	1.011655379	−0.2494239452

2b Statement. Use the values of y and dy/dt at $t = -0.04$ and $t = -0.07$ to construct a **Hermite cubic** $H(t)$ interpolating these values. Also find the polynomial $H'(t)$.

2b Solution. The easiest formula for the Hermite cubic interpolating polynomial is the one used in forming Bézier curves:

$$\begin{aligned} H(t) = {} & f(a)\left(\frac{t-b}{a-b}\right)^3 \\ & + 3\left(f(a) + \frac{b-a}{3}f'(a)\right)\left(\frac{t-b}{a-b}\right)^2\left(\frac{t-a}{b-a}\right) \\ & + 3\left(f(b) + \frac{a-b}{3}f'(b)\right)\left(\frac{t-b}{a-b}\right)\left(\frac{t-a}{b-a}\right)^2 \\ & + f(b)\left(\frac{t-a}{b-a}\right)^3 \end{aligned}$$

Substituting $a = -0.04$ and $b = -0.07$, we have $f(a) = 1.001706559$, $f(b) = 1.005469831$, $f(a) - 0.01 f'(a) = 1.002586423$, and $f(b) + 0.01 f'(b) = 1.003826164$. The resulting expression is

$$\begin{aligned} H(t) = {} & 1.001706559\left(\frac{t+0.07}{0.03}\right)^3 \\ & + 3(1.002586423)\left(\frac{t+0.07}{0.03}\right)^2\left(\frac{t+0.04}{-0.03}\right) \\ & + 3(1.003826164)\left(\frac{t+0.07}{0.03}\right)\left(\frac{t+0.04}{-0.03}\right)^2 \\ & + 1.005469831\left(\frac{t+0.04}{-0.03}\right)^3 \end{aligned}$$

The derivative has the general form

$$H'(t) = f'(a)\left(\frac{t-b}{a-b}\right)^2 + 2\big(3f[a,b] - f'(a) - f'(b)\big)\left(\frac{t-b}{a-b}\right)\left(\frac{t-a}{b-a}\right) + f'(b)\left(\frac{t-a}{b-a}\right)^2$$

where $f[a,b]$ is the **divided difference** $\big(f(b) - f(a)\big)/(b-a)$. For our special case, $f[-0.03, -0.07] = -.1254424000$, and $3f[-0.03, -0.07] - f'(-0.03) - f'(-0.07) = -.1239741110$, so that the derivative becomes

$$\begin{aligned} H'(t) = {} & -0.08798637078\left(\frac{t+0.07}{0.03}\right)^2 \\ & + 2(-.1239741110)\left(\frac{t+0.07}{0.03}\right)\left(\frac{t+0.04}{-0.03}\right) \\ & - 0.1643667182\left(\frac{t+0.04}{-0.03}\right)^2 \end{aligned}$$

2c Statement. Compare the values in the table and part (a) for $t = -0.05$ and $t = -0.06$ with the values of $H(-0.05)$, $H(-0.06)$, $H'(-0.05)$, and $H'(-0.06)$.

2c Solution. Here are the results (again, as computed in Maple)

$$\begin{aligned} f(-0.05) &= 1.002708001 & H(-0.05) &= 1.002708013 \\ f'(-0.05) &= -0.1124662414 & H'(-0.05) &= -.11246763 \\ f(-0.06) &= 1.003959161 & H(-0.06) &= 1.003959174 \\ f'(-0.06) &= -0.1379290161 & H'(-0.06) &= -.13792776 \end{aligned}$$

We notice a difference in the value of the function of about 10^{-8} as expected from a fourth order method with $h = -0.01$ and a difference in the valu of the derivative that is about 10^{-6}, indicating that it is a third order approximation. Since only the values of the function are computed by a fourth order method, this is reasonable.

End of workshop 8