# Mathematics 373 Workshop 6 Solutions

## Integration

## Fall 2003

**Problem 1.** On the interval $[-1, 1]$ and expressed in terms of averages, Simpson's rule has the form

$$\frac{1}{2}\int_{-1}^{1} f(t)\,dt = \frac{f(-1) + 4f(0) + f(1)}{6} - \frac{1}{180} f^{(4)}(\tau)$$

**1a Statement.** Check this formula using the function $f(t) = t^4$. Since this function has a constant fourth derivative, the error term must show the exact difference between the two other quantities.

**1a Solution.** If $f(t) = t^4$,

$$\frac{1}{2}\int_{-1}^{1} f(t)\,dt = \frac{1}{2} \cdot \left.\frac{t}{5}\right|_{-1}^{1} = \frac{1}{5}$$

and

$$\frac{f(-1) + 4f(0) + f(1)}{6} = \frac{1}{3}.$$

The error term is

$$-\frac{1}{180} f^{(4)}(\tau) = -\frac{24}{180} = -\frac{2}{15},$$

and it is easily checked that $1/5 = 1/3 - 2/15$.

**1b Statement.** To apply this to the interval $[x_0, x_1]$ write

$$x = \frac{x_1 + x_0}{2} + \frac{x_1 - x_0}{2} t,$$

which describes how to parameterize this interval with a variable $t$ that goes from $-1$ to $1$. Each term is modified differently by this change: in the integral, one has $dx = (x_1 - x_0)\,dt/2$; in the finite average, the values $t = -1$, $t = 0$, and $t = 1$ designate particular values of $x$; and in the error term, differentiating with respect to $t$ multiplies the derivative with respect to $x$ by $(x_1 - x_0)/2$. Show that these modifications lead to the formula (4.24) of the text when all notation is adjusted.

**1b Solution.** Since we are naming **functions**, which are the process of going from one quantity to another, it is convenient to introduce a new name $g(x)$ so that

$$g(x) = f\left(\frac{2x - x_0 - x_1}{x_1 - x_0}\right) \qquad (x_0 \le x \le x_1).$$

This is equivalent to

$$g\left(\frac{x_1 + x_0}{2} + \frac{x_1 - x_0}{2} t\right) = f(t) \qquad (-1 \le t \le 1)$$

Thus, $g(x) = f(t)$ with particular linear functions relating $x$ and $t$. Then, the averages are related by

$$\frac{1}{2}\int_{-1}^{1} f(t)\,dt = \frac{1}{x_1 - x_0}\int_{x_0}^{x_1} g(x)\,dx$$

and

$$\frac{f(-1) + 4f(0) + f(1)}{6} = \frac{g(x_0) + 4g(\frac{x_0+x_1}{2}) + g(x_1)}{6}.$$

For the error term, we note that

$$f'(t) = \frac{d}{dt}f(t) = \frac{d}{dx}g(x)\frac{dx}{dt} = \frac{x_1 - x_0}{2}g'(x)$$

since $x = (x + 1 + x_0)/2 + t(x + 1 - x_0)/2$. Each successive derivative introduces another factor of the constant $(x + 1 - x_0)/2$, so that the number of such factors is the same as the number of derivatives taken. In the notation of the text, $x + 1 - x_0 = 2h$, so we are led to

$$\frac{1}{2h}\int_{x_0}^{x_1} g(x)\,dx = \frac{g(x_0) + 4g(\frac{x_0+x_1}{2}) + g(x_1)}{6} - \frac{1}{180}h^4 g^{(4)}(\xi)$$

where $xi = (x + 1 + x_0)/2 + \tau(x + 1 - x_0)/2$. To get (4.24), multiply all terms by $2h$, and change the name of the function.

**1c Statement.** The **composite Simpson's rule** is constructed by dividing a given interval $[a, b]$ into n subintervals by choosing equally spaced points $a = x_0, x_1, \ldots, x_{n-1}, x_n = b$ and applying the basic Simpson's rule to each interval $[x_{k-1}, x_k]$. Give the details for $n = 2$.

**1c Solution.** Here $[a, b]$ is to be written as a union of two intervals $[x_0, x_1]$ and $[x_1, x_2]$, and the integral on each of these intervals will use a weighted average of values of the function at the endpoints and the midpoint of the interval. Thus, $x_0 = a$, $x_2 = b$ and $x_1 = (a + b)/2$ and the use of Simpson's rule on $[x_0, x_1]$ and $[x_1, x_2]$ will introduce the value of the function at the midpoint of the interval. Each $x_1 - x_{i-1}$ is $(b - a)/2$ and $2h$, so $h = (b - a)/4$ and

$$\int_a^b g(x)\,dx = \int_{x_0}^{x_1} g(x)\,dx + \int_{x_1}^{x_2} g(x)\,dx =$$

$$h\left(\frac{g(x_0) + 4g(\frac{x_0+x_1}{2}) + g(x_1)}{3} + \frac{g(x_1) + 4g(\frac{x_1+x_2}{2}) + g(x_2)}{3}\right) - \frac{h^5}{90}\big(g^{(4)}(\xi_1) + g^{(4)}(\xi_2)\big).$$

The value of $g(x_1)$ appears twice, so it could be written as one term with a coefficient of 2 instead of two terms with coefficients of 1. The sum of the two error terms could be written as twice an average error term, and this average error term involves a value of $g^{(4)}$ at some point in $[a, b]$.

**Notes.** The use of Simpson's rule requires consideration of the function at points other than the $x_k$ described in (c). Traditionally, those extra points are given equal treatment with the $x_k$ leading to formulas in which points with odd and even indices are treated differently. For this reason, your results may not look like those described in the textbook, but they really are the same.

If the composite Simpson's rule is considered as built from the simple Simpson's rule, the midpoints of the subintervals (those points whose function values have a coefficient of 4) play a different role than the other points.

## Problem 2.

**Romberg integration** uses extrapolation to produce higher order integration rules from lower order ones. We consider the first step in this extrapolation. The midpoint and trapezoidal rules on the standard intervals $[-1, 1]$ become

$$\frac{1}{2}\int_{-1}^{1} f(t)\,dt = f(0) + \frac{1}{6}f''(\tau) \qquad (M)$$

$$= \frac{f(-1)+f(1)}{2} - \frac{1}{3}f''(\tau) \qquad (T)$$

(corrected from original statement)

**2a Statement.** Check these formulas using the function $f(t) = t^2$. Since this function has a constant second derivative, the error term must show the exact difference between the two other quantities.

**2a Solution.** If $f(t) = t^2$, we have $f''(\tau) = 2$ and

$$\frac{1}{2}\int_{-1}^{1} f(t)\,dt = \frac{1}{3}$$

$$f(0) + \frac{1}{6}f''(\tau) = 0 + \frac{2}{6} = \frac{1}{3}$$

$$\frac{f(-1)+f(1)}{2} - \frac{1}{3}f''(\tau) = 1 - \frac{2}{3} = \frac{1}{3}$$

**2b Statement.** Construct the **composite trapezoidal rule** obtained by splitting the standard interval $[-1, 1]$ at 0. Express this result as a combination of the rules $(T)$ and $(M)$ on $[-1, 1]$.

**2b Solution.** The **average value** given by the composite trapezoidal rule describe in the statement is

$$\frac{1}{2}\left(\frac{f(-1)+f(0)}{2} + \frac{f(0)+f(1)}{2}\right) = \frac{1}{2}\,\frac{f(-1)+2f(0)+f(1)}{2} = \frac{1}{2}\left(\frac{f(-1)+f(1)}{2} + f(0)\right)$$

This is the average of the trapezoidal and midpoint rules.

**2c Statement.** Find another combination of $(T)$ and $(M)$ that causes the second order part of the error term to cancel out, allowing higher order parts to show through. You should recognize this expression.

**2c Solution.** The error term in $(T)$ is twice as large as, and opposite in sign to, the error term in $(M)$. So, $(2M + T)/3$ is an average of these quantities that will represent their common value with a smaller error term. The main term of this expression is

$$\frac{1}{3}\left(2f(0) + \frac{f(-1)+f(1)}{2}\right) = \frac{f(-1)+4f(0)+f(1)}{6}.$$

This is Simpson's rule, which we have already found to have a fourth order error term.

**Notes.** The symmetric nature of these formulas leads to an asymptotic formula for the error term in which only even powers of the step size appear. When extrapolation is used on this result to remove the fourth order error term, the result will have a sixth order error term. Continued extrapolation leads to the Romberg scheme.

End of workshop 6