MATH 252:01 — SAMPLE HOUR EXAM 2

Sketches are omitted from this sample.

1. Let $\mathbf{A} = \begin{pmatrix} 4 & 5 \\ -10 & -11 \end{pmatrix}$ in the following.

(a) Find the eigenvalues of $\mathbf{A}$ and **find an eigenvector** belonging to each eigenvalue.

(b) Find the general solution of the system $\dfrac{d\mathbf{Y}(t)}{dt} = \mathbf{A}\mathbf{Y}(t)$, using your results of **(a)**.

(c) Using your results of **(b)**, find the solution of $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ for which $\mathbf{Y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

2. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -26 & -2 \end{pmatrix}$ in the following.

(a) **Find** the (complex!) eigenvalues of $\mathbf{A}$.

(b) **Find** the (complex) eigenvector belonging to one (your choice) of the eigenvalues you found in **(a)**.

(c) **Give the elements of the basis of $\mathbb{R}^2$ that is involved** in solving the differential equation $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ using the eigenvector you found in **(a)**.

3. The eigenvalues of $\mathbf{A} = \begin{pmatrix} 0 & -5 \\ 1 & -4 \end{pmatrix}$ are $\lambda = -2 \pm i$, and a complex eigenvector belonging to $-2+i$ is $\begin{pmatrix} 2+i \\ 1 \end{pmatrix}$. Use that information to answer the following.

(a) **What is the complex solution** of $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ associated with the complex eigenvalue and eigenvector given above?

(b) Using the complex solution you gave in **(a)** above, **give two linearly independent $\mathbb{R}^2$-valued solutions** of $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$.

(c) **State** whether the origin is a source, a sink or a center for the system $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$.

4. **(a)** **Find the general solution** of the homogeneous second-order linear differential equation

$$\frac{d^2y}{dt^2} - 3\,\frac{dy}{dt} + 2y = 0\ .$$

(b) **Find a particular solution** of the (inhomogeneous) second-order linear DE

$$\frac{d^2y}{dt^2} - 3\,\frac{dy}{dt} + 2y = 20\,\sin 2t\ .$$

You may use either a complex-exponential "good guess" method or a method of undetermined coefficients (substitute $y = A\cos\omega t + B\sin\omega t$ for a suitably chosen ω, compare coefficients on the r. h. and l. h. sides of the DE, then solve the resulting equations for A and B).

5. The equation of motion for a certain undamped harmonic oscillator (weight-and spring system, with $y(t)$ measuring the displacement of the weight from the equilibrium position $y = 0$) is

$$\frac{d^2y}{dt^2} = -25y\ .$$

We wish to add damping to the system in the form of a force directed contrary to the motion and proportional to the velocity of the oscillating mass, with proportionality constant $b > 0$, so that the equation of motion becomes

$$\frac{d^2y}{dt^2} = -b\,\frac{dy}{dt} - 25y\ .$$

(a) **What value of** b will be **just large enough** to prevent the system from oscillating?

(b) If the system is left undamped (so $b = 0$) in the equation above, and if the weight is deflected $+2$ units from equilibrium and released at time zero (*i.e.*, $y(0) = 2$, $y'(0) = 0$), **what will be the position of the weight at time** $t = 1$? (Answer this in terms of values of exp, cos and sin if you can; use a calculator if you feel you must.)

6. On the next page of this examination are four `Maple` sketches of solution curves of various second-order homogeneous linear DEs, all of the form

$$\frac{d^2y}{dt^2} + p\,\frac{dy}{dt} + qy = 0\ ,$$

and satisfying one of the pairs of initial conditions

$$y(0) = 1, \quad y'(0) = 0 \qquad \text{or} \qquad y(0) = 0, \quad y'(0) = 1\ .$$

Label each sketch with three things: the **sign of** p, the **sign of the discriminant** $p^2 - 4q$ that occurs under the radical in the quadratic-formula solution of the characteristic equation, and the **initial conditions** that the solution satisfies (you need only make the choice between the two pairs of initial conditions given above).

7. On the next (and thus last) page of this examination are four `Maple` sketches of direction fields. Below are seven first-order linear homogeneous 2×2 systems. **Label each sketch with the number** (Roman numeral) **of the system to which it corresponds**. There is *at most* one system corresponding to any particular sketch.

(i) $$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -4 \end{pmatrix} \mathbf{Y}(t)$$

(ii) $$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 5 \\ 1 & 2 \end{pmatrix} \mathbf{Y}(t)$$

(iii) $$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix} \mathbf{Y}(t)$$

(iv) $$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 5 \\ -2 & 0 \end{pmatrix} \mathbf{Y}(t)$$

(v) $$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & -2 \\ 5 & -2 \end{pmatrix} \mathbf{Y}(t)$$

(vi) $$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & -5 \\ 2 & 0 \end{pmatrix} \mathbf{Y}(t)$$

(vii) $$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -5 \\ 2 & -1 \end{pmatrix} \mathbf{Y}(t)$$