FIRST-ORDER LINEAR SYSTEMS WHOSE COEFFICIENT MATRICES ARE SKEW-SYMMETRIC

These systems turn up often enough in applications to justify looking at them closely. I started on them in class but didn't do enough with them because of time pressure. These notes fill in the details for people who may need them in later courses. This is not examination material.

0. *Linear-algebra preliminaries.*

In order to read all but the last § of these notes, one should be familiar with the properties of the dot and cross products in $\mathbb{R}^3$ as exposed in most[1] calculus textbooks. One should also know that there is a dot product in $\mathbb{R}^n$ analogous to the familiar one in $\mathbb{R}^3$; if one thinks of the elements of $\mathbb{R}^n$ as $n \times 1$ column matrices and identifies the 1×1 matrices with $\mathbb{R}$, then one can write the dot product in the following way:

$$\text{If } \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{then } \mathbf{x} \bullet \mathbf{y} = \mathbf{y}^T \mathbf{x} = \mathbf{x}^T \mathbf{y} = \sum_{i=1}^{n} x_i y_i \tag{0.1}$$

where $\mathbf{M}^T$ denotes the transpose of a given matrix $\mathbf{M}$—the matrix found from the original $\mathbf{M}$ by "interchanging rows and columns." The dot product interacts with matrix multiplication: if $\mathbf{A} = (a_{ij})$ is an $n \times n$ matrix applied to $\mathbf{x}$, then the i-th coördinate of $\mathbf{Ax}$ is the sum $\sum_{j=1}^{n} a_{ij} x_j$. Dotting this with $\mathbf{y}$ and computing the sum in reverse order gives

$$(\mathbf{Ax}) \bullet \mathbf{y} = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} a_{ij} x_j \right] y_i = \sum_{j=1}^{n} x_j \left[\sum_{i=1}^{n} a_{ij} y_i \right] \tag{0.2}$$

so that $\mathbf{x}$ is being dotted with a vector that is determined by multiplying $\mathbf{y}$ by a certain matrix—one obviously related to $\mathbf{A}$. However, because the summation $\sum_{i=1}^{n} a_{ij} y_i$ is being taken over the "wrong" index, the matrix being applied to $\mathbf{y}$ is not $\mathbf{A}$, but rather its transpose $\mathbf{A}^T$. Thus we have the relation

$$(\mathbf{Ax}) \bullet \mathbf{y} = \mathbf{x} \bullet (\mathbf{A}^T \mathbf{y}) , \tag{0.3}$$

sometimes expressed by saying that "when the matrix goes across the dot product it gets transposed." Note that if one expresses the dot product in terms of transposing the vectors considered as $n \times 1$ matrices, then **(0.2)** and **(0.3)** are neatly expressed by

$$(\mathbf{Ax}) \bullet \mathbf{y} = \mathbf{y}^T \mathbf{Ax} = (\mathbf{A}^T \mathbf{y})^T \mathbf{x} = \mathbf{x} \bullet (\mathbf{A}^T \mathbf{y}) \tag{0.4}$$

and both follow from the fact that transposition reverses the order of matrix multiplication (and the fact that the dot product is symmetric in its arguments).

1. *Skew-symmetric first-order systems: first principles.*

First-semester linear-algebra courses sometimes get as far as investigating the eigenvalue/eigenvector behavior of **symmetric** matrices, those for which $\mathbf{A}^T = \mathbf{A}$. From the standpoint of differential equations, the **skew-symmetric** matrices—those for which $\mathbf{A}^T = -\mathbf{A}$, are every bit as interesting, because of the following proposition. First we need a small

Lemma: The following properties are equivalent for $n \times n$ real matrices $\mathbf{A}$:

(1) For every $\mathbf{y} \in \mathbb{R}^n$, $(\mathbf{Ay}) \bullet \mathbf{y} = -\mathbf{y} \bullet \mathbf{Ay}$.

(2) For every $\mathbf{y} \in \mathbb{R}^n$, $(\mathbf{Ay}) \bullet \mathbf{y} = 0$.

[1] In J. Stewart, *Calculus: Early Transcendentals*, 3rd ed., see Ch. 11, p. 664 ff.; 4th ed., Ch. 12, p. 782 ff.

(3) $\mathbf{A} = -\mathbf{A}^T$, that is, $\mathbf{A}$ is skew-symmetric.

Proof. **(1)** $\Rightarrow$ **(2)**: The dot product is symmetric in its arguments, so $(\mathbf{A}\mathbf{y}) \bullet \mathbf{y} = \mathbf{y} \bullet (\mathbf{A}\mathbf{y})$ holds for every $\mathbf{A}$ and $\mathbf{y}$; putting that relation together with **(1)**, we find that $(\mathbf{A}\mathbf{y}) \bullet \mathbf{y} = 0$ must hold for every $\mathbf{y} \in \mathbb{R}^n$. **(2)** $\Rightarrow$ **(3)**: replacing $\mathbf{y}$ by $\mathbf{x} + \mathbf{y}$, we see that for every $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$

$$\begin{aligned}
&\mathbf{A}(\mathbf{x}+\mathbf{y}) \bullet (\mathbf{x}+\mathbf{y}) = 0 \\
&(\mathbf{A}\mathbf{x}) \bullet \mathbf{x} + (\mathbf{A}\mathbf{y}) \bullet \mathbf{x} + (\mathbf{A}\mathbf{x}) \bullet \mathbf{y} + (\mathbf{A}\mathbf{y}) \bullet \mathbf{y} = 0 \\
&0 + (\mathbf{A}\mathbf{y}) \bullet \mathbf{x} + (\mathbf{A}\mathbf{x}) \bullet \mathbf{y} + 0 = 0 \\
&\mathbf{A}^T\mathbf{x} \bullet \mathbf{y} = \mathbf{x} \bullet (\mathbf{A}\mathbf{y}) = (\mathbf{A}\mathbf{y}) \bullet \mathbf{x} = -(\mathbf{A}\mathbf{x}) \bullet \mathbf{y} \\
&[(\mathbf{A}^T + \mathbf{A})\mathbf{x}] \bullet \mathbf{y} = 0 \qquad \textbf{(1.1)}
\end{aligned}$$

and because **(1.1)** holds for *every* $\mathbf{y} \in \mathbb{R}^n$, one must[2] have $(\mathbf{A}^T + \mathbf{A})\mathbf{x} = \mathbf{0}$; but since this holds for *every* $\mathbf{x} \in \mathbb{R}^n$ (including all the standard basis vectors), the matrix $\mathbf{A}^T + \mathbf{A}$ must be the zero matrix, or $\mathbf{A}^T = -\mathbf{A}$. **(3)** $\Rightarrow$ **(1)**: If $\mathbf{A}^T = -\mathbf{A}$ then $(\mathbf{A}\mathbf{y}) \bullet \mathbf{y} = \mathbf{y} \bullet \mathbf{A}^T\mathbf{y} = -\mathbf{y} \bullet \mathbf{A}\mathbf{y}$ holds for every $\mathbf{y} \in \mathbb{R}^n$.

Proposition: The first-order linear homogenous system $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ has the property that all its solutions have **constant distance from the origin**—$\|\mathbf{Y}(t)\| \equiv \|\mathbf{Y}(0)\|$—if and only if the matrix of coefficients satisfies $\mathbf{A}^T = -\mathbf{A}$; that is, if and only if $\mathbf{A}$ is a skew-symmetric matrix.

Proof. The dot product of $\mathbf{Y}(t)$ with itself is the length-squared of the vector $\mathbf{Y}(t)$. Consequently[3]

$$\begin{aligned}
\frac{d}{dt}\|\mathbf{Y}(t)\|^2 &= \frac{d}{dt}(\mathbf{Y}(t) \bullet \mathbf{Y}(t)) = \mathbf{Y}'(t) \bullet \mathbf{Y}(t) + \mathbf{Y}(t) \bullet \mathbf{Y}'(t) \\
&= \mathbf{A}\mathbf{Y}(t) \bullet \mathbf{Y}(t) + \mathbf{Y}(t) \bullet \mathbf{A}\mathbf{Y}(t) = (\mathbf{A}\mathbf{Y}(t)) \bullet \mathbf{Y}(t) + (\mathbf{A}^T\mathbf{Y}(t)) \bullet \mathbf{Y}(t) \\
&= [(\mathbf{A} + \mathbf{A}^T)\mathbf{Y}(t)] \bullet \mathbf{Y}(t)\ . \qquad \textbf{(1.2)}
\end{aligned}$$

From the last line of **(1.2)** it follows that if $\mathbf{A}^T = -\mathbf{A}$, then the derivative of $\|\mathbf{Y}(t)\|^2$ is zero for all t, so the length of $\mathbf{Y}(t)$ is a constant—and the constant must be $\mathbf{Y}(0)$. Conversely, if the length of $\mathbf{Y}(t)$ is constant so the derivative of its length-squared is zero, then—since $\mathbf{Y}(0)$ could be any vector in $\mathbb{R}^n$—the matrix $\mathbf{A}$ must have the property that $[(\mathbf{A} + \mathbf{A}^T)\mathbf{y}] \bullet \mathbf{y} = 0$, or equivalently that $(\mathbf{A}\mathbf{y}) \bullet \mathbf{y} = -\mathbf{y} \bullet (\mathbf{A}\mathbf{y})$ holds for all vectors $\mathbf{y} \in \mathbb{R}^n$. By the lemma above, $\mathbf{A}$ must be skew-symmetric.

Skew-symmetric matrices have no nonzero real eigenvalues; in fact, their eigenvalues are always pure-imaginary. The reason is simple: if $\mathbf{A}$ is skew-symmetric and $\mathbf{x}$ is a *real* eigenvector belonging to a *real* eigenvalue λ of $\mathbf{A}$, then $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, so $\lambda\mathbf{x} \bullet \mathbf{x} = (\mathbf{A}\mathbf{x}) \bullet \mathbf{x} = \mathbf{x} \bullet (\mathbf{A}^T\mathbf{x}) = -\mathbf{x} \bullet (\mathbf{A}\mathbf{x}) = -\lambda\mathbf{x} \bullet \mathbf{x}$. If $\mathbf{x} \neq \mathbf{0}$ so its length-squared $\|\mathbf{x}\|^2 = \mathbf{x} \bullet \mathbf{x} > 0$, we can divide by it to get $\lambda = -\lambda$ and therefore $\lambda = 0$. On the other hand, it is known[4] that a symmetric matrix always has only real eigenvalues. If $\mathbf{A} = -\mathbf{A}^T$ then the matrix $\mathbf{A}^T\mathbf{A} = -\mathbf{A}^2$ is symmetric; thus if λ is a *complex* eigenvalue of $\mathbf{A}$ with *complex* eigenvector $\mathbf{z}$ we have $\mathbf{A}^2\mathbf{z} = \mathbf{A}[\mathbf{A}\mathbf{z}] = \mathbf{A}[\lambda\mathbf{z}] = \lambda^2\mathbf{z}$ and λ^2 is therefore real. But then one must have $\lambda^2 < 0$ or λ itself would have been real; so λ is a square root of a negative number and is therefore a pure imaginary.

If the dimension n of the space is odd, a (real) skew-symmetric $n \times n$ matrix always has zero as a real eigenvalue (with a real eigenvector). The reason is that (in all dimensions) the determinant of any matrix is the same as that of its transpose, but the determinant of the matrix $-\mathbf{I}$ is $(-1)^n$ (the product of the (-1)'s on its diagonal), which is $+1$ or -1 according to whether n is even or odd respectively. So if n is odd we have

$$\begin{aligned}
\det(\mathbf{A}) &= \det(\mathbf{A}^T) = \det(-\mathbf{A}) = \det((-\mathbf{I})(\mathbf{A})) \\
&= (-1)^{\text{odd}}\det(\mathbf{A}) = -\det(\mathbf{A}) \\
\det(\mathbf{A}) &= 0 \qquad \textbf{(1.3)}
\end{aligned}$$

[2] If a vector is orthogonal to every vector in $\mathbb{R}^n$, then in particular it is orthogonal to itself; but then, since its dot product with itself is its length-squared, it must have zero length; therefore it must be the zero vector. Or: let $\mathbf{y}$ run through the standard basis vectors, and thus observe that every coördinate of a vector orthogonal to all vectors must be zero.

[3] Apropos differentiation of the dot product of two vector-valued functions, see Stewart, 3rd ed. p. 709, box (5); 4th ed. p. 846, box (3).

[4] See, for example, K. Hoffman & R. Kunze, *Linear Algebra*, 2nd ed. (1971), p. 312, Theorem 15.

and so there is some $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$ with $\mathbf{A}\mathbf{x} = \mathbf{0}$.

2. *Skew-symmetry in low dimensions.*

This is all more concrete than it may look at first, and the cases of dimension $n = 2$ and $n = 3$ give insight into the general situation.

In the case $n = 2$, a skew-symmetric matrix has to have the form $\mathbf{A} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$, where $\omega \in \mathbb{R}$. We know all about these: the general solution of $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ for such an $\mathbf{A}$ is given by

$$\mathbf{Y}(t) = \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix} \mathbf{Y}(0) \tag{2.1}$$

and the trajectory of every point is the circle centered at the origin that passes through the point; the moving point traverses the circle at angular velocity ω; the eigenvalues of $\mathbf{A}$ are $\pm i\omega$.

In the case $n = 3$, we can realize all the skew-symmetric 3×3 matrices in a familiar form.

Proposition For any vector $\mathbf{a} = (a_1, a_2, a_3)^T \in \mathbb{R}^3$, the matrix of the linear transformation $\mathbf{x} \mapsto \mathbf{a} \times \mathbf{x}$ (where "$\times$" is the usual cross product) relative to the standard basis is the skew-symmetric matrix $\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$, and every skew-symmetric 3×3 matrix is the matrix of such a linear transformation for an $\mathbf{a} \in \mathbb{R}^3$ that is uniquely determined by the matrix.

Proof. By taking "$\mathbf{x}$" to be each vector of the standard basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in order, one easily checks that the matrix of $\mathbf{x} \mapsto \mathbf{a} \times \mathbf{x}$ is as advertised, so we omit the details. Given a skew-symmetric 3×3 matrix $\mathbf{A}$, we can find one and only one $(a_1, a_2, a_3)^T \in \mathbb{R}^3$ so that the—say—above-diagonal entries of the matrix of $\mathbf{x} \mapsto \mathbf{a} \times \mathbf{x}$ match those of $\mathbf{A}$; but then, since both matrices are skew-symmetric, *all* of the entries match; so every 3×3 skew-symmetric matrix is obtained in this way for a unique $\mathbf{a} \in \mathbb{R}^3$.

We can use the "$\mathbf{a}$" of the proposition to help understand the less transparent "$\mathbf{A}$." For example, since $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any $\mathbf{a} \in \mathbb{R}^3$, we see where the eigenvector belonging to zero comes from: it's $\mathbf{a}$ itself. The pure-imaginary eigenvalues are also related to $\mathbf{a}$: the determinant computation that gives us the characteristic polynomial

$$\begin{aligned} \det(\lambda \mathbf{I} - \mathbf{A}) &= \det \begin{bmatrix} \lambda & a_3 & -a_2 \\ -a_3 & \lambda & a_1 \\ a_2 & -a_1 & \lambda \end{bmatrix} \\ &= \lambda(\lambda^2 + a_1^2) + a_3(a_3\lambda - a_1 a_2) + a_2(a_3 a_1 + a_2 \lambda) \\ &= \lambda^3 + (a_1^2 + a_2^2 + a_3^2)\lambda = \lambda(\lambda^2 + \|\mathbf{a}\|^2) \end{aligned} \tag{2.2}$$

shows—of course—that $\lambda = 0$ is an eigenvalue, and also shows that the other two eigenvalues of $\mathbf{A}$ are $\pm i\omega$ where $\omega = \|\mathbf{a}\|$. Let $\mathbf{v}_3 = \dfrac{1}{\|\mathbf{a}\|}\mathbf{a}$, the unit vector pointing in the same direction as $\mathbf{a}$. We shall show that from the standpoint of "looking down from the tip of $\mathbf{v}_3$," things look about as they did in $\mathbb{R}^2$; namely, the solutions of $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ hold the $\mathbf{v}_3$-axis fixed and rotate the plane orthogonal to it at angular velocity $\omega = \|\mathbf{a}\|$. To see this, let $\mathbf{w}$ be a *complex* eigenvector[5] of $\mathbf{A}$ belonging to $i\omega$, and—taking its real and imaginary parts—write it (uniquely) in the form $\mathbf{w} = \mathbf{v}_1 - i\mathbf{v}_2$, where the two vectors $\mathbf{v}_1$, $\mathbf{v}_2 \in \mathbb{R}^3$. (The

[5] At this stage of our development of these ideas, there is considerable ambiguity in picking this eigenvector; we shall adjust its length (divide it by a positive constant) later.

perhaps-unexpected minus sign inserted in the imaginary part will make the computations come out more nicely.) Applying $\mathbf{A}$ to both sides of the relation $\mathbf{w} = \mathbf{v}_1 - i\mathbf{v}_2$ produces, just as in the two-dimensional case with which we are so familiar,

$$\begin{aligned} \mathbf{A}\mathbf{v}_1 - i\mathbf{A}\mathbf{v}_2 = \mathbf{A}\mathbf{w} = i\omega\mathbf{w} = i\omega(\mathbf{v}_1 - i\mathbf{v}_2) = \omega\mathbf{v}_2 + i\omega\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_1 = \omega\mathbf{v}_2 \qquad \text{and} \qquad \mathbf{A}\mathbf{v}_2 = -\omega\mathbf{v}_1 \ . \end{aligned} \tag{2.3}$$

This already tells us that the matrix of $\mathbf{A}$ relative to the basis $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$ has the form $\begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$—the upper left 2×2 matrix looks familiar—but things will get even better. Indeed, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal to each other, have the same length, and are both orthogonal to $\mathbf{v}_3$. They are orthogonal to each other because $\mathbf{A}\mathbf{x} \bullet \mathbf{x} = 0$ for any vector $\mathbf{x}$ due to the skew-symmetry of $\mathbf{A}$, and therefore

$$\omega(\mathbf{v}_2 \bullet \mathbf{v}_1) = \omega\mathbf{v}_2 \bullet \mathbf{v}_1 = \mathbf{A}\mathbf{v}_1 \bullet \mathbf{v}_1 = 0 \ ; \tag{2.4}$$

but since $\omega = \|\mathbf{a}\| > 0$, one must have $\mathbf{v}_2 \bullet \mathbf{v}_1 = 0$. They have the same length because one can divide $\omega^2 \neq 0$ out of both sides of

$$\begin{aligned} \omega^2(\mathbf{v}_2 \bullet \mathbf{v}_2) = \omega^2\mathbf{v}_2 \bullet \mathbf{v}_2 &= \mathbf{A}\mathbf{v}_1 \bullet \mathbf{A}\mathbf{v}_1 \\ &= (-\mathbf{A}^2\mathbf{v}_1) \bullet \mathbf{v}_1 = (\omega^2\mathbf{v}_1) \bullet \mathbf{v}_1 = \omega^2(\mathbf{v}_1 \bullet \mathbf{v}_1) \ ; \\ \|\mathbf{v}_2\|^2 = \mathbf{v}_2 \bullet \mathbf{v}_2 = \mathbf{v}_1 \bullet \mathbf{v}_1 = \|\mathbf{v}_1\|^2 \ . \end{aligned} \tag{2.5}$$

It now appears that, by dividing the original complex eigenvector $\mathbf{w}$ by the common value of $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| > 0$, we could have arranged to have $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$, so we shall assume that that has been done. Both $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal to $\mathbf{v}_3$, since $\mathbf{v}_3$ points in the same direction as $\mathbf{a}$, each $\|\mathbf{a}\|\mathbf{v}_i = \pm\mathbf{A}\mathbf{v}_j = \pm\mathbf{a} \times \mathbf{v}_j$, and $\mathbf{a} \times \mathbf{x}$ is orthogonal to $\mathbf{a}$ for any $\mathbf{x} \in \mathbb{R}^3$. So $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$ is an orthonormal basis of $\mathbb{R}^3$; moreover, it is **positively oriented** in the sense that it obeys the same $\times$-multiplication table that $\{\mathbf{i},\, \mathbf{j},\, \mathbf{k}\}$ does:

$$\begin{aligned} &\mathbf{v}_3 \times \mathbf{v}_1 = \frac{\mathbf{a}}{\|\mathbf{a}\|} \times \mathbf{v}_1 = \frac{\mathbf{A}\mathbf{v}_1}{\|\mathbf{a}\|} = \mathbf{v}_2 \\ &\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v}_1 \times (\mathbf{v}_3 \times \mathbf{v}_1) = (\mathbf{v}_1 \bullet \mathbf{v}_1)\mathbf{v}_3 - (\mathbf{v}_1 \bullet \mathbf{v}_3)\mathbf{v}_1 = \mathbf{v}_3 \\ &\mathbf{v}_2 \times \mathbf{v}_3 = -\mathbf{v}_3 \times \mathbf{v}_2 = -\frac{1}{\|\mathbf{a}\|}\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 \ , \end{aligned} \tag{2.6}$$

the second line of **(2.6)** involving the vector triple product identity.[6] The fact that the basis $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$ is an orthonormal basis makes the change-of basis relation particularly simple, because if $\mathbf{Q} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \vdots & \vdots & \vdots \end{bmatrix}$ is the matrix whose columns are these three vectors, then $\mathbf{Q}^{-1} = \mathbf{Q}^T$ (verification immediate). We know from what we have just demonstrated that

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{equivalently} \quad \mathbf{A} = \mathbf{Q}\begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{Q}^{-1} \ . \tag{2.7}$$

[6] See Stewart, 3rd ed. p. 686, 4th ed. p. 807, box (8) Theorem #6.

It now follows from what we know about $\exp\left(t\begin{bmatrix}0 & -\omega\\ \omega & 0\end{bmatrix}\right)$ that

$$e^{t\mathbf{A}} = \mathbf{Q}\begin{bmatrix}\cos\omega t & -\sin\omega t & 0\\ \sin\omega t & \cos\omega t & 0\\ 0 & 0 & 1\end{bmatrix}\mathbf{Q}^T\ , \qquad (\mathbf{2.8})$$

i.e., that the matrix $e^{t\mathbf{A}}$ rotates $\mathbb{R}^3$ around the axis $\mathbf{v}_3$, at an angular velocity of $\omega = \|\mathbf{a}\|$, in a positive direction when viewed from the tip of the vector $\mathbf{a}$ (or of its normalized version $\mathbf{v}_3$). Since the plane spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane orthogonal to $\mathbf{a}$, one may view the solution of $\mathbf{Y}' = \mathbf{AY}$, or $\mathbf{Y}' = \mathbf{a}\times\mathbf{Y}(t)$, geometrically as follows: given an initial-position vector $\mathbf{Y}_0$, resolve it into its components along and orthogonal to $\mathbf{a}$, calling them (say) $\mathbf{Y_a}$ and $\mathbf{Y}_\perp$. Then $\mathbf{Y}(t) = \mathbf{Y_a} + \mathbf{Y}_\perp(t)$: the component along the axis $\mathbf{a}$ does not change, but the orthogonal component is rotated in the positive direction, with a rotation of $\omega t = \|\mathbf{a}\|$ (radians) having been achieved at time t.

This relation among (1) the cross product, (2) skew-symmetric matrices, (3) first-order homogeneous linear systems of DEs that leave lengths invariant and (4) rotations about a fixed axis at constant angular velocity, is the reason that the cross product turns up so frequently in the discussions of rotation, torque, and rotational inertia in classical mechanics. For example, the definitions of **torque** in the form $\mathbf{r}\times\mathbf{F}$ and of **moment of momentum** as $\mathbf{r}\times\left(m\,\dfrac{d^2\mathbf{r}}{dt^2}\right)$ yield such elegant results precisely because the cross product is an "infinitesimal rotation about the axis given by one factor": the exponential series tells us that in a small time interval Δt the moving point that started at $\mathbf{Y}_0$ at time $t = 0$ and is now at $\mathbf{Y}(t) = e^{t\mathbf{A}}\mathbf{Y}_0$ will be at $\mathbf{Y}(\Delta t + t) = e^{(\Delta t+t)\mathbf{A}}\mathbf{Y}_0 = e^{\Delta t\mathbf{A}}e^{t\mathbf{A}}\mathbf{Y}_0 = e^{\Delta t\mathbf{A}}\mathbf{Y}(t) \sim \mathbf{Y}(t) + \Delta t\,\mathbf{a}\times\mathbf{Y}(t)$ a "short time" Δt later, but (because of the distance-preserving property) the distance of the moving point from the origin will not have changed. (The physical quantities occur in analyzing the motion of a rigid body—**rigidity** means that distances do not change with time.)

One should give a concrete computed example, so suppose $\mathbf{a} = (6, 6, 3)^T$; then $\|\mathbf{a}\| = 9$ and $\mathbf{v}_3 = (2/3, 2/3, 1/3)^T$. The eigenvalues of $\mathbf{A}$ are $\pm 9i$. To find a complex eigenvector belonging to $9i$ we have to solve

$$\begin{bmatrix}9i & 3 & -6\\ -3 & 9i & 6\\ 6 & -6 & 9i\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix} = \mathbf{0}\ . \qquad (\mathbf{2.9})$$

In view of the geometry of the situation we should be able to take $\mathbf{v}_1$ to be any unit vector orthogonal to $\mathbf{a}$, so we try for a solution of **(2.9)** whose real part[7] is the unit vector $(1/3, -2/3, 2/3)^T$. Thus we need to solve for *real* numbers x, y, z with

$$\begin{bmatrix}9i & 3 & -6\\ -3 & 9i & 6\\ 6 & -6 & 9i\end{bmatrix}\begin{bmatrix}1/3 - xi\\ -2/3 - yi\\ 2/3 - zi\end{bmatrix} = \mathbf{0}\ . \qquad (\mathbf{2.10})$$

The first equation becomes $3i + 9x - 2 - 3iy - 4 + 6iz = 0$, so $x = 2/3$ and $y - 2z = 1$. The second equation becomes $-1 + 3ix - 6i + 9y + 4 - 6iz = 0$, so $y = -1/3$ and $3x - 6 - 6z = 0$, which gives $z = -2/3$ (and these are consistent with the earlier equation connecting y and z). This is a unit vector as predicted, and it is straightforward to verify that the ordered basis $\{(1/3, -2/3, 2/3)^T, (2/3, -1/3, -2/3)^T, (2/3, 2/3, 1/3)^T\}$

[7] Obviously there is some chicanery going on in the choice of these vectors; I wanted to be able to give pretty numbers to avoid turning the reader off (to a greater degree than was absolutely necessary, anyway). The vector $(1/9)\mathbf{a}$ in this example is one element of a well-known orthonormal basis that consists of vectors all of whose coördinates are rational; I am making sure that I get the other two elements of that basis (heh heh).

is orthonormal and has the correct ×-multiplication table. The interested reader might find it amusing to compute $e^{t\mathbf{A}}$ explicitly by writing out **(2.8)** above with this $\mathbf{Q} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$ and with $\omega = 9$. Differentiation at time $t = 0$ should then give the same result as crossing with $\mathbf{a}$.

3. *Skew-symmetry in higher dimensions.*

Much of what we did above—as long as we don't need the cross product, which is not available[8] in higher dimensions, carries over to the case of $n \times n$ skew-symmetric matrices for $n > 3$, at least in the case in which $\mathbf{A}$ has n distinct eigenvalues. If n is even these would necessarily come in distinct conjugate pure-imaginary pairs except for zero—and for even n we have thus ruled out the possibility that zero is an eigenvalue, since the null space would have to have dimension at most 1 but its dimension would have to be even. If n is odd then zero must be an eigenvalue, and it will be the only real eigenvalue. If $i\omega \neq 0$ is a pure-imaginary eigenvalue of skew-symmetric $\mathbf{A}$ and $\mathbf{w}$ is a complex eigenvector belonging to it, then the argument that established **(2.3)**, **(2.4)** and **(2.5)** above is valid—of course we cannot say $\omega = \|\mathbf{a}\|$, since there is no $\mathbf{a}$ in this context—but the plane spanned by the real and imaginary parts of $\mathbf{w}$ (which in the context of §2 were $\mathbf{v}_1$ and $\mathbf{v}_2$) is mapped onto itself by $\mathbf{A}$, those two vectors are orthogonal and have the same length (which we can adjust to be 1), and the matrix of $\mathbf{A}$ acting on that plane will be the familiar $\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$ with respect to that basis. If $i\omega$ and $i\varphi$ are distinct eigenvalues (with $\varphi \neq -\omega$, of course) then their planes are orthogonal to one another, because if $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vectors constructed using ω and $\mathbf{u}_1$ and $\mathbf{u}_2$ the ones constructed using φ, then since $\mathbf{A}^2\mathbf{v}_1 = -\omega^2\mathbf{v}_1$ and $\mathbf{A}^2\mathbf{u}_1 = -\varphi^2\mathbf{u}_1$, the two vectors $\mathbf{v}_1$ and $\mathbf{u}_1$ are eigenvectors of the symmetric matrix $-\mathbf{A}^2$ that belong to different eigenvalues, and it is well known and easy to prove[9] that they must be orthogonal to one another. But the choice of the complex eigenvectors belonging to $i\omega$ and $i\varphi$ was arbitrary, and the real part of $\mathbf{w}$ is the imaginary part of $i\mathbf{w}$, so $\mathbf{v}_2$ and $\mathbf{u}_1$ are just as orthogonal as $\mathbf{v}_1$ and $\mathbf{u}_1$ were. One can thus write $\mathbb{R}^n$ as an "orthogonal direct sum" of planes (2-dimensional subspaces), each of which is spanned by two orthogonal vectors of length 1 with respect to which $\mathbf{A}$ acting on that subspace has a matrix $\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$. This thus gives us an orthonormal basis of $\mathbb{R}^n$ with respect to which $\mathbf{A}$ has a matrix of a form

$$\begin{bmatrix} \begin{bmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{bmatrix} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} 0 & -\omega_2 \\ \omega_2 & 0 \end{bmatrix} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & [0] \end{bmatrix} . \qquad (\mathbf{3.1})$$

and $e^{t\mathbf{A}}$ will similarly have a matrix in which the 2×2 blocks down the main diagonal have the form

$$\begin{bmatrix} \cos\omega_j t & -\sin\omega_j t \\ \sin\omega_j t & \cos\omega_j t \end{bmatrix} \mathbf{Y}(0) . \qquad (\mathbf{3.2})$$

where the ω_j range through the nonzero eigenvalues $\pm i\omega_j$ of $\mathbf{A}$ and the "axis" represented by the one-dimensional block at the bottom of the matrix in **(3.1)** will be present if and only if n is odd (recall that

[8] There are infinitely many cross products available in $\mathbb{R}^7$, but they lack many of the properties of the familiar cross product in 3 dimensions. Only the 3- and 7-dimensional spaces possess cross products at all. See, for example, the obscure paper *The scarcity of cross products on Euclidean spaces*, Am. Math. Monthly 74 (1967), pp. 188–194.

[9] See Hoffman & Kunze, *ibid.*

we are assuming only one complex eigen-direction for each complex eigenvalue and so the existence of a unique eigen-direction belonging to 0 makes the dimension odd). Just as in the 2- and 3-dimensional cases things "rotate in planes," but because of the high dimension they can rotate at different angular velocities in different planes.

The most satisfactory treatment of these matters—and a real understanding of them—requires a full-scale use of complex-inner-product methods, but lets us drop the hypothesis that the eigenvalues are distinct. One uses the **complex dot product** or **complex inner product** on $\mathbb{C}^n$ defined by

$$\text{If } \mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \quad \text{then } \mathbf{z} \bullet \mathbf{w} = \mathbf{w}^* \mathbf{z} = \overline{\mathbf{z}^* \mathbf{w}} = \sum_{j=1}^{n} z_j \overline{w}_j$$

$$\text{with length-squared} \quad \|\mathbf{z}\|^2 = \mathbf{z} \bullet \mathbf{z} = \sum_{j=1}^{n} |z_j|^2 \ . \tag{3.3}$$

Here the overbar (for example, $\overline{w}$) denotes the complex conjugate, and the **adjoint** of a matrix $\mathbf{C} = (c_{ij})$, written $\mathbf{C}^*$, is its **conjugate transpose**, so $\mathbf{C}^* = (\overline{c}_{ji})$. The effect of this definition is to replace the (evident) linearity of the operation of taking transposes by **conjugate** linearity:

$$(\alpha \mathbf{A} + \beta \mathbf{B})^* = \overline{\alpha} \mathbf{A}^* + \overline{\beta} \mathbf{B}^* \ . \tag{3.4}$$

We put up with such inconveniences because the (crucial!) effect of inserting the conjugations is to make the complex dot product of $\mathbf{z} = (z_1, \ldots, z_n)^T \in \mathbb{C}^n$ with itself be the same number that would result if we were to write $z_j = x_j + iy_j$ for each complex coördinate (so $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ where each vector $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$) and then take the sum of the squares of their components, added together:

$$\|\mathbf{z}\|^2 = \sum_{j=1}^{n} z_j \overline{z}_j = \sum_{j=1}^{n} |z_j|^2 = \sum_{j=1}^{n} (x_j^2 + y_j^2) \tag{3.5c}$$

$$= \sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{n} y_j^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \ . \tag{3.5r}$$

Thus, as far as distance relations are concerned, $\mathbb{C}^n$ with this notion of length is the same as $\mathbb{R}^{2n}$. **Orthogonality** of two vectors in $\mathbb{C}^n$ continues to be defined by the property of having a dot product—now the *complex* dot product—equal to zero. The symmetry of the real dot product is replaced by **conjugate**-symmetry:

$$\mathbf{z} \bullet \mathbf{w} = \overline{\mathbf{w} \bullet \mathbf{z}} \tag{3.6}$$

and since $\overline{0} = 0$ this makes the relation "$\mathbf{z}$ is orthogonal to $\mathbf{w}$" continue to be a symmetric one. The relations **(0.3)** and **(0.4)** above are replaced by

$$(\mathbf{A}\mathbf{z}) \bullet \mathbf{w} = \mathbf{z} \bullet (\mathbf{A}^* \mathbf{w}) \tag{3.7}$$

and

$$(\mathbf{A}\mathbf{z}) \bullet \mathbf{w} = \mathbf{w}^* \mathbf{A} \mathbf{z} = (\mathbf{A}^* \mathbf{w})^* \mathbf{z} = \mathbf{z} \bullet (\mathbf{A}^* \mathbf{w}) \ . \tag{3.8}$$

The notion of a symmetric matrix is replaced by that of a **conjugate-symmetric** matrix, usually called a **self-adjoint** or sometimes **Hermitian** matrix; these are the matrices (c_{ij}) for which $c_{ij} = \overline{c}_{ji}$, or equivalently $\mathbf{C} = \mathbf{C}^*$. These complex matrices have the same property with respect to the complex dot product that symmetric real matrices have with respect to the real dot product: all their eigenvalues of a self-adjoint matrix are real (though the vectors that belong to them may still be complex), and they possess orthonormal bases of eigenvectors. For proofs of these assertions, see Hoffman & Kunze, *ibid*.

Conjugate linearity is nicer than it looks for matrices $\mathbf{A}$ that are skew-symmetric and have real entries, because the relation

$$(-i\mathbf{A})^* = \overline{-i}\,\mathbf{A}^* = i\,(-\mathbf{A}) = -i\mathbf{A} \tag{3.9}$$

turns the difficult skew-symmetric real matrix $\mathbf{A}$ into an easy-to-handle complex self-adjoint matrix $-i\mathbf{A}$ that can be diagonalized by a (complex-)orthonormal basis of (complex) eigenvectors. Multiplying $-i\mathbf{A}$ by i turns it back into $\mathbf{A}$ but simply multiplies its (complex) diagonalized form by i's on the diagonal; so $\mathbf{A}$ is also diagonalized, by the same (still complex) basis. The non-real—and therefore nonzero—eigenvalues of $\mathbf{A}$ come in pure-imaginary conjugate pairs, because the characteristic polynomial of $\mathbf{A}$ is a real polynomial; and if $-i\mathbf{A}\mathbf{z} = \omega\mathbf{z}$—equivalently, $\mathbf{A}\mathbf{z} = i\omega\mathbf{z}$—then conjugating everything in sight gives $i\mathbf{A}\overline{\mathbf{z}} = \omega\overline{\mathbf{z}}$—equivalently, $\mathbf{A}\mathbf{z} = -i\omega\mathbf{z}$. (Note that this implies that the [real] eigenvalues of $-i\mathbf{A}$ must have come in positive-negative pairs, except for zero.)

By reëxamining the diagonalization process for the self-adjoint matrix $-i\mathbf{A}$, we can see how to imitate the construction we made in the 3-dimensional case. Let $\omega > 0$ be a positive (real) eigenvalue of $-i\mathbf{A}$, and find a (complex-)orthogonal[10] basis of the eigenspace $\{\mathbf{z} \in \mathbb{C}^n : -i\mathbf{A}\mathbf{z} = \omega\mathbf{z}\}$; if its (complex) dimension is ℓ, we could index the elements of the basis as $\{\mathbf{z}_1, \ldots, \mathbf{z}_\ell\}$. Then for each $1 \le k \le \ell$, we have $\mathbf{A}\mathbf{z}_k = i\omega\mathbf{z}_k$ and therefore—by conjugating everything in sight—$\mathbf{A}\overline{\mathbf{z}}_k = -i\omega\overline{\mathbf{z}}_k$, or $-i\mathbf{A}\overline{\mathbf{z}}_k = -\omega\overline{\mathbf{z}}_k$. Thus each $\overline{\mathbf{z}}_k$ is an eigenvector (of the self-adjoint matrix $-i\mathbf{A}$) that belongs to $-\omega \ne \omega$, and so each $\overline{\mathbf{z}}_k$ is orthogonal to all the $\mathbf{z}_j$'s, $1 \le j \le \ell$. In particular, $\overline{\mathbf{z}}_k \perp \mathbf{z}_k$. If we write that fact out in (real) coördinates, writing $\mathbf{z}_k = (x_1 - iy_1, \ldots, x_n - iy_n)^T$, we find that because the second factor in a complex dot product is conjugated inside the definition of the dot product, the relation

$$0 = \mathbf{z}_k \bullet \overline{\mathbf{z}}_k = \sum_{j=1}^{n} z_j^2 = \sum_{j=1}^{n}(x_j - iy_j)^2 = \sum_{j=1}^{n}(x^2 - y^2) - i\sum_{j=1}^{n}(2x_jy_j) \tag{3.10c}$$

holds. Relation **(3.10c)** is a complex equation equivalent to the two real equations

$$\sum_{j=1}^{n} x_j^2 = \sum_{j=1}^{n} y^2 \qquad \text{and} \qquad \sum_{j=1}^{n} x_jy_j = 0\,. \tag{3.10r}$$

These in turn are actually real-dot-product relations: if we set $\mathbf{x}_k = (x_1, \ldots, x_n)^T = \mathrm{Re}\,[\mathbf{z}_k]$ and $\mathbf{y}_k = (y_1, \ldots, y_n)^T = -\mathrm{Im}\,[\mathbf{z}_k]$, then **(3.10r)** can be written in the equivalent real-vector form

$$\|\mathbf{x}_k\| = \|\mathbf{y}_k\| \qquad \text{and} \qquad \mathbf{x}_k \bullet \mathbf{y}_k = 0\,. \tag{3.10v}$$

(Compare **(2.4)** and **(2.5)** above in the 3-dimensional case!) If we look at the subspaces of $\mathbb{C}^n$ and $\mathbb{R}^n$ spanned by $\{\mathbf{x}_k, \mathbf{y}_k\}$, then, we see that the discussion above has established the following things:

(1) The 2-complex-dimensional subspace of $\mathbb{C}^n$ spanned by $\{\mathbf{x}_k, \mathbf{y}_k\}$ is the same as the 2-complex-dimensional subspace of $\mathbb{C}^n$ spanned by $\{\mathbf{z}_k, \overline{\mathbf{z}}_k\}$; moreover, $\{\mathbf{x}_k, \mathbf{y}_k\}$ is a complex-orthogonal basis of it because it is a real-orthogonal set. The relations

$$\begin{aligned} -i\mathbf{A}[\mathbf{x}_k - i\mathbf{y}_k] &= \omega\mathbf{x}_k - i\omega\mathbf{y}_k \\ \mathbf{A}\mathbf{x}_k - i\mathbf{A}\mathbf{y}_k = \mathbf{A}[\mathbf{x}_k - i\mathbf{y}_k] &= \omega\mathbf{y}_k + i\omega\mathbf{x}_k \\ \mathbf{A}\mathbf{x}_k = \omega\mathbf{y}_k \qquad \mathbf{A}\mathbf{y}_k &= -\omega\mathbf{x}_k \end{aligned} \tag{3.11}$$

are equivalent to the fact that the matrix of $\mathbf{A}$ acting on this subspace relative to this basis is $\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$.

(2) The 2-real-dimensional subspace of $\mathbb{R}^n$ spanned by $\{\mathbf{x}_k, \mathbf{y}_k\}$ has $\{\mathbf{x}_k, \mathbf{y}_k\}$ as a real-orthogonal basis. If $\mathbf{z}_k$ was normalized to make either (and therefore both) of $\mathbf{x}_k$ and/or $\mathbf{y}_k$ have length 1, then $\{\mathbf{x}_k, \mathbf{y}_k\}$ is a

[10] But not necessarily *orthonormal*; we shall want to pick this basis in such a way that the real and imaginary parts of its elements are real vectors of norm 1, as we shall see in what follows.

real orthonormal basis of this subspace. The matrix of $\mathbf{A}$ acting on this subspace, relative to this basis, is the same as the matrix of $\mathbf{A}$ seen in **(1)** above.

(3) For distinct indices $1 \le j \ne k \le \ell$, the 2-complex-dimensional subspace of $\mathbb{C}^n$ spanned by $\{\mathbf{x}_j, \mathbf{y}_j\}$ is complex-orthogonal to the 2-complex-dimensional subspace of $\mathbb{C}^n$ spanned by $\{\mathbf{x}_k, \mathbf{y}_k\}$, because the former is spanned by $\{\mathbf{z}_j, \overline{\mathbf{z}}_j\}$ and the latter by $\{\mathbf{z}_k, \overline{\mathbf{z}}_k\}$: $\mathbf{z}_j \perp \mathbf{z}_k$ because their indices are different and $\overline{\mathbf{z}}_j \perp \overline{\mathbf{z}}_k$ follows from that relation by conjugating everything in sight; $\mathbf{z}_j \perp \overline{\mathbf{z}}_k$ because the two vectors belong to different eigenvalues, and the same is true in the case of $\overline{\mathbf{z}}_j \perp \mathbf{z}_k$.

(4) For distinct indices $1 \le j \ne k \le \ell$, the 2-real-dimensional subspace of $\mathbb{R}^n$ spanned by $\{\mathbf{x}_j, \mathbf{y}_j\}$ is real-orthogonal to the 2-real-dimensional subspace of $\mathbb{R}^n$ spanned by $\{\mathbf{x}_k, \mathbf{y}_k\}$, because—as we just saw in **(3)**—their bases belong to complex-orthogonal subspaces of $\mathbb{C}^n$, and the real and complex inner products give the same value when applied to real vectors.

(5) If the construction just given is applied to an eigenvalue of $-i\mathbf{A}$ different from the ω used above, then all the resulting vectors are (complex- or real-)orthogonal to those obtained from ω in that construction. Their complex-orthogonality comes from the fact that they all belong to eigenvalues different from ω and $-\omega$, and the real-orthogonality follows again from the fact that the real and complex inner products give the same value when applied to real vectors.

From what we have done so far it is clear that we can find orthonormal pairs of real vectors $\{\mathbf{x}_\iota, \mathbf{y}_\iota\}$, with the pairs mutually orthogonal, such that the matrix of $\mathbf{A}$ on the real (or complex) subspace spanned by $\{\mathbf{x}_\iota, \mathbf{y}_\iota\}$ is $\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$. We can pick one such pair for each eigenvalue $\omega > 0$ of $-i\mathbf{A}$, counting the eigenvalues according to their multiplicity as roots of the characteristic polynomial of $-i\mathbf{A}$, and the complex subspace spanned by each such pair will be the orthogonal sum of a one-complex-dimensional subspace of $\mathbb{C}^n$ generated by and eigenvector belonging to ω and a one-complex-dimensional subspace of $\mathbb{C}^n$ generated by an eigenvector belonging to $-\omega$. If $-i\mathbf{A}$—or equivalently $\mathbf{A}$—has a nontrivial null space (as it must have if n is odd), we can pick an orthogonal basis of that also, and we can pick the complex basis to consist of real vectors of length 1 since if $\mathbf{A}[\mathbf{x} + i\mathbf{y}] = \mathbf{0}$ then the fact that $\mathbf{A}$ has real entries implies that $\mathbf{A}\mathbf{x} = \mathbf{0}$ and $\mathbf{A}\mathbf{y} = \mathbf{0}$ separately. Thus we have shown that there is a real orthonormal basis of $\mathbb{R}^n$ (which is also an orthogonal basis of $\mathbb{C}^n$) with respect to which $\mathbf{A}$ has a matrix of the form

$$\begin{bmatrix} \begin{bmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{bmatrix} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} 0 & -\omega_2 \\ \omega_2 & 0 \end{bmatrix} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \end{bmatrix}. \qquad \mathbf{(3.12)}$$

That true phoenix, the interested reader, could try working out the 3×3 case using the 3×3 matrix we analyzed at the end of §2 above as an example of this general development. Multiply that matrix by i, find eigenvectors of $i\mathbf{A}$ belonging to ± 9, and watch what happens using the complex dot product.