Math 252 — Spring 2000
Exact Solution of the Predator-Prey System

Introduction. In Section 2.1 of the textbook, the system

$$\begin{aligned}\frac{dR}{dt} &= 2R - 1.2RF \\ \frac{dF}{dt} &= -F + 0.9RF\end{aligned} \tag{P}$$

is studied from several different viewpoints. The aim is to collect evidence that the solutions follow closed curves in the RF-plane (the *phase plane* of the system). Such a strong conclusions should have a more convincing proof. These notes aim to provide such a proof. Along the way, we will sketch a proof that the curves in the phase plane representing the solution of an autonomous system do not cross except possibly at equilibrium points of the system. A more powerful approach is given in Section 2.4, but the case of autonomous systems in two variables can be done using only results that you have already seen.

Slope fields, vector fields and direction fields. When we were studying equations of the form $dy/dt = f(t, y)$ in a (t, y)-plane, we drew lines of slope $f(t, y)$ at an array of points (t, y). These lines were usually scaled to have the same length, just because it gave a prettier picture. In the text, short line segments were used to illustrate such a **slope field**, but *Maple* added an arrow at the end with the larger value of t.

In the case of an autonomous system, the right sides of the equations give the components of a vector that shows the derivative with respect to t of the dependent variables of the system along a solution curve. This is a velocity vector of a particle whose motion is governed by the system. A collection of these velocity vectors on an array of points in the phase plane gives a **velocity field**. Just as for single autonomous equations, we will see that each solution determines a family of solutions that differ only by a translation of t, so a picture in a phase space coordinatized only by the dependent variables of the system gives useful information about the solutions of the equation. If these velocity vectors are drawn, they give some indication of the speed with which the curves in the phase plane are being drawn.

Just as in the study of curves in *Multivariable Calculus*, it is common to suppress the details of a parameterization of a curve and represent the direction of a curve at a point by its *unit tangent vector* which is the velocity vector scaled to have length 1. The **equilibrium points** are special since the functions which have those points as constant values satisfy the equation. For all other points, the velocity vector is nonzero, and hence has a well-defined direction. Using these unit vectors gives a **direction field** in the phase plane. The direction field shows the nature of the solution curves of the equation, except that it has lost information about how fast we are drawing the curve at different points.

Rabbits and foxes. We now concentrate on the system (P). Since $dR/dt = 1.2R(5/3 - F)$, R will be an increasing function of t as long as we remain in the region where $R > 0$ and $F < 5/3$. The other regions bounded by the lines where $dR/dt = 0$ will behave similarly. The fact that R is an increasing function of t guarantees that there is an inverse function that allows us to locally measure time by counting the number of rabbits. There will be a complicated relation between R and t, and this silly clock will fail when F reaches $5/3$, but it is still useful for theoretical purposes. In particular, we can eliminate t and consider F as a function of R as long as $F < 5/3$.

This gives the equation

$$\frac{dF}{dR} = \frac{-F + 0.9RF}{2R - 1.2RF} = \frac{F}{2 - 1.2F} \cdot \frac{-1 + 0.9R}{R}. \tag{Q}$$

Equation (Q) is separable, so

$$\int \frac{2-1.2F}{F}\,dF = \int \frac{-1+0.9R}{R}\,dR$$
$$\int \left(\frac{2}{F}-1.2\right)dF = \int \left(\frac{-1}{R}+0.9\right)dR$$
$$2\ln F - 1.2F = -\ln R + 0.9R + C.$$

The function of F on the left is increasing for $F < 5/3$ and decreasing for $F > 5/3$ (obviously! consider how it was found), while the function on the right has a minimum at $R = 10/9$. This means that only values of $C < \ln 250 - 4\ln 3 - 3 \approx -1.873$ will arise. For each such C there will an interval of values of F less than $5/3$ for which we can solve to obtain two values for R. For the limiting value, $F = 5/3$ the values of R give the endpoints of the interval for which a value of F can be found.

Now, we move to the interval where $F > 5/3$. The same conclusions are found in this interval, and the same values of R corresponding to $F = 5/3$ are found. In other words, choosing one value of R for $F = 5/3$ determines the same value for the other value of R with $F = 5/3$, whether we are looking at the upper or lower region. This shows that the solutions follow closed curves in the (R, F)-plane.

In those examples in which the solutions show a spiral behavior, the only difference will be that the formulas for matching endpoints will differ on the two sides of curve where the dF/dR fails to exist.

Taking the fox's viewpoint. One could also use F to measure time. To do this, this lines where $dF/dt = 0$ need to be excluded. These are different lines from those used above, but the analysis is the same. It is useful to note that, except for the equilibrium points where both derivatives are zero, all points in the plane lie in the interior of one of regions where we can interpret the direction field of the equation as a slope field. This means that the closed orbits for this system really are the smooth curves that the numerical methods show. Unfortunately, the accumulated error in numerical methods would suggest that the orbits might be spirals instead of closed orbits, so some analysis is required to give the true behavior of the equation. The technique of *linearization* will allow us to identify examples where there must be spiraling behavior, but closed orbits turn out to be special.