LINEARIZED VERSIONS OF §2.1, §2.2 SYSTEMS

0. *Taylor Series in Several Variables.*

We want to understand how "linearization near an equilibrium point" in several variables—the analogue of the test for sources and sinks by using the sign of the derivative of the r. h. s. of a single autonomous D. E. developed in §1.6, particularly pp. 84–86—can be done in the several-dependent-variable case that we are examining in Ch. 2. In one variable, the basic idea was this: if the r. h. s. of the D. E.

DE $$\frac{dy}{dt} = f(y)$$

is zero at y_0, *i.e.*, if $y(t) \equiv y_0$ is an equilibrium solution of the D. E., then for y near y_0 we have the Taylor approximation of the r. h. s.

$$\begin{aligned} f(y) &= f(y_0) + \frac{f'(y_0)}{1!}(y - y_0) + \frac{f''(y_0)}{2!}(y - y_0)^2 + \cdots \\ &\approx f'(y_0) \cdot (y - y_0) \end{aligned}$$

(remember, $f(y_0) = 0$ by definition of equilibrium) and so the solutions of the D. E. "near" the equilibrium solution will tend to behave like solutions of the D. E.

LIN DE $$\frac{dy}{dt} = a \cdot (y - y_0)$$

where $a = f'(y_0)$. This D. E. has the explicitly computable solutions

$$y(t) = y_0 + C \cdot e^{at} \qquad C = \text{some constant}$$

and as $t \to +\infty$ these tend toward the line $y = y_0$ if $a = f'(y_0) < 0$—y_0 is a **sink**—while they tend away from the line $y = y_0$ if $a = f'(y_0) > 0$—y_0 is a **source**. The key to this analysis is the Taylor-series[1] approximation to the function $f(y)$ for values of y near y_0.

To do the same analysis for systems of more-than-one D. E., like the fox-and-rabbit systems of §2.1, we have to do Taylor-series approximations in more than one variable. This is a type of approximation that everybody is *supposed* to learn in first-semester sophomore calculus.[2] If $\Phi(x, y)$ is a function of two variables defined near a point $P_0 = (x_0, y_0)$, the Taylor expansion takes the form

$$\begin{aligned} \Phi(x,y) = \Phi(x_0,y_0) &+ \left.\frac{\partial \Phi}{\partial x}\right|_{P_0}(x - x_0) + \left.\frac{\partial \Phi}{\partial y}\right|_{P_0}(y - y_0) \\ &+ \frac{1}{2!}\left\{ \left.\frac{\partial^2 \Phi}{\partial x^2}\right|_{P_0}(x - x_0)^2 + 2\left.\frac{\partial^2 \Phi}{\partial x \partial y}\right|_{P_0}(x - x_0)(y - y_0) + \left.\frac{\partial^2 \Phi}{\partial y^2}\right|_{P_0}(y - y_0)^2 \right\} + \cdots \end{aligned}$$

but—fortunately—for our purposes it will suffice to take only the "first-degree Taylor polynomial" as a good approximation to our functions:

$$\Phi(x,y) \approx \Phi(x_0,y_0) + \left.\frac{\partial \Phi}{\partial x}\right|_{P_0}(x - x_0) + \left.\frac{\partial \Phi}{\partial y}\right|_{P_0}(y - y_0)\ .$$

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[1] Actually, it's the simplest kind of Taylor series: it's just a tangent-line approximation to the function $f(y)$.

[2] If you learned calculus at Rutgers recently, you will most likely have been cheated out of this information by the ineffable James Stewart, *Calculus: Early Transcendentals*, 3rd or 4th ed. Fortunately, you can get by in the present course by simply using the "tangent-plane" or "total-differential" approximation to the surface. In the 3rd edition, Stewart gives this information in such places in Chapter 12 as formula (7), p. 768 or the boxed Theorem (10) of p. 771. In the 4th edition, this material is moved to Chapter 14, formula [10], p. 913 and the boxed Theorem (1) of the previous edition has been split into a definition—[7] on p. 911—and a theorem—[8] on the next page (912). Probably responding to the protests of people who try to teach mathematics from his book, Stewart has made the second-degree Taylor polynomial in two variables into a "Discovery Project" on p. 950. Should you have occasion to think about the partial differential equations of mathematical physics, you may want to make friends with the Taylor series in more-than-one variable from a serious calculus text. Some would regard knowledge of this material as a survival matter.

1. *The First Predator-Prey System.*

This is the system discussed on pp. 140–144 of the text. We want to see how an analytical approach to the system might suggest behavior like that depicted by the computer-approximate solutions that produced Figures 2.4 and 2.5. The interesting equilibrium point for the system is at $R = 10/9$, $F = 5/3$. The Taylor expansions of the r. h. sides of the two equations of the system about these points are

$$2R - \frac{12}{10}RF = 0 + [2 - \frac{12}{10}\frac{5}{3}](R - \frac{10}{9}) + [-\frac{12}{10}\frac{10}{9}](F - \frac{5}{3}) - \frac{12}{10}(R - \frac{10}{9})(F - \frac{5}{3})$$

and

$$-F + \frac{9}{10}RF = 0 + [\frac{9}{10}\frac{5}{3}](R - \frac{10}{9}) + [-1 + \frac{9}{10}\frac{10}{9}](F - \frac{5}{3}) + \frac{9}{10}(R - \frac{10}{9})(F - \frac{5}{3})$$

respectively. If we introduce the new coördinates r, f defined by $r = R - 10/9$ and $f = F - 5/3$—*i.e.*, center the new coördinates on the equilibrium point—then $dR/dt = dr/dt$ and $dF/dt = df/dt$ and the equations become (after simplification of the Taylor expansions on the last two set-off lines)

SYS
$$\frac{dr}{dt} = -\frac{4}{3}f + \frac{6}{5}rf$$
$$\frac{df}{dt} = \frac{3}{2}r + \frac{9}{10}rf\ .$$

The equilibrium point is $(r, f) = (0, 0)$ in the new coördinate system, and for solutions that stay "near" this point—whatever "near" may mean—we can hope that the solutions will stay "near" the solutions of the **linearized system**

LIN SYS
$$\frac{dr}{dt} = -\frac{4}{3}f$$
$$\frac{df}{dt} = \frac{3}{2}r$$

which we may hope to analyze in a manner similar to the way we used LIN DE to analyze sources and sinks of DE in the single-equation case.

Because we do not yet have a general way to solve linear systems (as we do single linear equations), we must now do something *ad hoc* which will only fit into a general context later. If we differentiate the first equation in LIN SYS and use the second equation in the result, we get

$$\frac{d^2r}{dt^2} = -\frac{4}{3}\frac{df}{dt}$$
$$\frac{df}{dt} = \frac{3}{2}r$$
$$\frac{d^2r}{dt^2} = -\frac{4}{3}\frac{3}{2}r = -2r\ .$$

Thus $r(t)$ is a function whose second derivative is (up to the multiplier 2) its negative. Sines and cosines have this property; we can get the multiplier 2 to come out in two differentiations by using the argument $\sqrt{2}t$; and so it is routine to check that for any constant A the functions

$$r(t) = A\cos(\sqrt{2}t)$$
$$f(t) = \frac{3\sqrt{2}}{4}A\sin(\sqrt{2}t)$$

satisfy LIN SYS. (We got $f(t)$ by forcing it to be $-\frac{3}{4}\frac{dr}{dt}$.) Everybody knows that $t \mapsto (\rho\cos\omega t, \rho\sin\omega t)$ is the way to parametrize a circle of radius ρ centered at the origin; if instead one uses $t \mapsto (a\cos\omega t, b\sin\omega t)$, the moving point traverses the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with semi-axes a and b. We thus see that the moving point $t \mapsto (r(t), f(t))$ traverses an ellipse with semi-axes A and $\frac{3\sqrt{2}}{4} A$ centered at the equilibrium point. (After we finish Ch. 3 we shall know that the general solution of LIN SYS will have the form

$$r(t) = A\cos(\sqrt{2}(t - t_0))$$
$$f(t) = \frac{3\sqrt{2}}{4} A\sin(\sqrt{2}(t - t_0))$$

so in fact we have the most general trajectories—only the "time at which the clock starts" might be different.) The pictures of the (computer-approximately-) true solutions of the equations SYS in Figures 2.4 and 2.5, while they are seriously bent out of elliptical shape, nonetheless represent closed curves circling the equilibrium point, very much like our solutions of the simplified LIN SYS, and we may hope that this "family resemblance" is a generalization to systems of the source-and-sink classification for single equations that was obtained by linearizing them around equilibrium points.

2. *The Predator-Prey System With Limited Food for Prey.*

Let us try the same approach with the "Modified Predator-Prey Model" of p. 144 ff. Here the system is

$$\frac{dR}{dt} = 2R\left(1 - \frac{R}{2}\right) - \frac{6}{5}RF$$
$$\frac{dF}{dt} = -F + \frac{9}{10}RF\ ,$$

and the textbook finds an interesting equilibrium point at $(\frac{10}{9}, \frac{20}{27})$. Introducing $r = R - \frac{10}{9}$ and $f = F - \frac{20}{27}$ as before, we find that the system of equations becomes (you should check the details!)

$$\frac{dr}{dt} = \frac{dR}{dt} = -\frac{10}{9}r - \frac{4}{3}f - r^2 - \frac{6}{5}rf \approx -\frac{10}{9}r - \frac{4}{3}f$$
$$\frac{df}{dt} = \frac{dF}{dt} = \frac{2}{3}r + \frac{9}{10}rf \approx \frac{2}{3}r\ .$$

We can use the same approach to the "approximate" linearized system that we used before: we differentiate the first equation and combine the result with the second equation to give

$$\frac{d^2r}{dt^2} = -\frac{10}{9}\frac{dr}{dt} - \frac{4}{3}\frac{df}{dt}$$
$$\frac{df}{dt} = \frac{2}{3}r$$
$$\frac{d^2r}{ds^2} + \frac{10}{9}\frac{dr}{dt} + \frac{8}{9}r = 0\ .$$

For a reason which will become clearer later, we seek a solution in the form $r(t) = e^{-10t/18} \cdot \varphi(t)$. Plugging this into the second-order equation above we find after simplification, that $\varphi(t)$ must satisfy the D. E.

$$e^{-5t/9} \cdot \left[\varphi''(t) + \frac{47}{81}\varphi(t)\right] = 0\ .$$

Evidently if we can find $\varphi(t)$ for which $\varphi''(t) = -\frac{47}{81}\varphi(t)$ we may hope to have found $r(t)$, and in fact $\varphi(t) = A\cos\left(\frac{\sqrt{47}}{9}t\right)$ will do it.[3] Thus for each choice of A the two functions

$$r(t) = Ae^{-5t/9}\cos\left(\frac{\sqrt{47}}{9}t\right)$$
$$f(t) = -\frac{5}{6}r(t) - \frac{3}{4}r'(t) = Ae^{-5t/9}\left[\frac{-5}{12}\cos\left(\frac{\sqrt{47}}{9}t\right) + \frac{\sqrt{47}}{9}\sin\left(\frac{\sqrt{47}}{9}t\right)\right]$$

[3] You might want to check the reasoning, which is the same as it was in the preceding case.

give solutions of this system, and again we shall eventually find it possible to show that, up to a time shift, these are the only solutions. If the factor $e^{-5t/9}$ were not present, it would be possible (by completing-the-square techniques, among other ways) to show that the moving point $(r(t), f(t))$ with coördinates given by

$$(r(t), f(t)) = \left(\cos\left(\frac{\sqrt{47}}{9}\,t\right), \frac{-5}{12}\cos\left(\frac{\sqrt{47}}{9}\,t\right) + \frac{\sqrt{47}}{9}\sin\left(\frac{\sqrt{47}}{9}\,t\right)\right)$$

moves in an ellipse (with axes that are not parallel to the coördinate axes) centered on zero. The presence of the exponentially-decreasing factor $e^{-5t/9}$, however, makes the moving point $(r(t), f(t))$ spiral in to $(0, 0)$ as $t \to +\infty$, which is behavior very much like that exhibited by the computer-generated approximations to true solutions that are depicted in Figures 2.6 and 2.7: after some initial fairly large oscillations, the populations quite rapidly approach an equilibrium point at which the supplies of cabbages, rabbits and foxes will oscillate only slightly with time.

3. *The Vibrating Spring.*

Here the equation we start with is linear. There are no yv-terms like the RF-terms we encountered with rabbits and foxes, so we do not have to find an equilibrium (or at least we don't have to work very hard: the only equilibrium solution finds the mass at rest at $(0, 0)$), Taylor-expand around it, and throw away the terms of higher degree: there are no terms of higher degree. There is nothing to add in these notes to the treatment in the text. We can, however, complicate life by making the model more realistic: everyone observes that springs don't go on vibrating forever, but rather vibrate with smaller amplitude as time passes and they lose energy to their surroundings. Thus the $F = ma$ equation might reasonably have the form

$$\begin{aligned} ma &= F = -bv - ky \\ m\,\frac{d^2y}{dt^2} &= -b\,\frac{dy}{dt} - ky \end{aligned}$$

where the new term, which corresponds to a force proportional (with proportionality-constant b) to velocity but directed oppositely to it, shows a force resisting the motion of the spring-loaded mass in its environment.[4] As a first-order system in the yv phase plane this equation would become

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{k}{m}\,y - \frac{b}{m}v\ , \end{aligned}$$

a trifle more complicated than the system envisioned on pp. 146–151 of the text. We shall find, however, that it yields to the same kind of attack. To keep the numbers easy, suppose that m, k and b are so chosen that the $F = ma$ equation becomes

$$\frac{d^2y}{dt^2} = -2\,\frac{dy}{dt} - 5\,y\ .$$

If we seek a solution in the form $y(t) = e^{-t}\varphi(t)$, we shall see that $\varphi(t)$ would have to satisfy the D. E.

$$\begin{aligned} e^{-t}\varphi'' - 2e^{-t}\varphi' + e^{-t}\varphi &= -2\left[e^{-t}\varphi' - e^{-t}\varphi\right] - 5e^{-t}\varphi \\ \varphi'' &= (2 - 5 - 1)\varphi = -4\varphi \end{aligned}$$

and again—with everything much like the situation with the rabbits and foxes—we see trigonometric solutions $\varphi(t) = A\cos 2t$. (It is not difficult to show that the most general solution is $\varphi(t) = A\cos 2(t - t_0)$; again, all that is lacking is a time shift.) The corresponding solutions of the original equation are then

$$y(t) = Ae^{-t}\cos(2t)\ ;$$

[4] This assemblage is the *damped linear harmonic oscillator*.

we see **damped harmonic motion**, whose oscillations die out exponentially-fast as $t \to +\infty$. In the yv-phase plane things resemble the rabbits-and-foxes pictures even more closely: we have

$$\begin{aligned} y(t) &= Ae^{-t}\cos 2t \\ v(t) &= Ae^{-t}\left[-\cos 2t - 2\sin 2t\right] \end{aligned}$$

and it is not difficult to show that the path $t \mapsto (\cos 2t, -\cos 2t - 2\sin 2t)$ is an ellipse with center zero being traversed in a negative direction. The effect of the "exponential decay factor" is then to make the moving point in the yv-plane spiral in toward the equilibrium point $(0,0)$; what one sees is much like what happened with the rabbits and foxes, although here (for no real reason—it's just a sign convention) the spiral paths are traversed in a clockwise direction.

4. *§2.1 Exercises.*

By the time you *have* to read this, we should have linearized these equations in class. Again, it is to be emphasized that you are not supposed to understand the details of this method now; it's scheduled for systematic coverage when we get to §5.1 of the textbook. It's just that one hates to say "turn to page n, class" in class . . . .

(i) The equilibrium points of

$$\begin{aligned} \frac{dx}{dt} &= 10x\left(1 - \frac{x}{10}\right) - 20xy \\ \frac{dy}{dt} &= -5y + \frac{xy}{20} \end{aligned}$$

are obtained by setting the r. h. sides of these equations equal to zero and solving the simultaneous equations in two unknowns that result; it is routine to find that the solutions are $(0,0)$ (rather trivially), $(10,0)$, and $(100,-9/2)$. It is easy to linearize these equations at $(0,0)$, since that amounts to just considering the terms of degree 0 and 1; we get the linearized system

$$\begin{aligned} \frac{dx}{dt} &= 10x \\ \frac{dy}{dt} &= -5y \end{aligned}$$

which is **decoupled** (see p. 175 ff.): the two populations are independent of each other in this approximation, with $x(t) = x(0)e^{10t}$ and $y(t) = y(0)e^{-5t}$. Note that we may expect this equilibrium point to behave (approximately) like a source for x and a sink for y, provided that $(x(0), y(0))$ is sufficiently close to $(0,0)$, and that seems intuitively reasonable.

If we expand the r. h. sides around the equilibrium point $(10,0)$ but only keep the linear part, we get

$$\begin{aligned} \frac{dX}{dt} = \frac{dx}{dt} &= -10X - 200y - X^2 - 20yX \approx -10X - 200y \\ \frac{dy}{dt} &= -\frac{9}{2}y + \frac{1}{20}Xy \approx -\frac{9}{2}y \end{aligned}$$

where $X = x - 10$ (the coördinate centered at the equilibrium point; since the equilibrium value of y is 0, we do not have to transform that coördinate). Because the equation in y does not involve x, we can simply solve it: $y' = -\frac{9}{2}y$ is a linear equation whose solution is $y(t) = y(0)e^{-9t/2}$. Plugging this into the equation for X gives us

$$\begin{aligned} \frac{dX}{dt} &= -10X - 200 \cdot y(0)e^{-9t/2} \\ \frac{d}{dt}\left[e^{10t}X(t)\right] &= -200 \cdot y(0)e^{11t/2} \\ e^{10t}X(t) &= -\frac{400}{11}y(0)e^{11t/2} + C \\ X(t) &= -\frac{400}{11}y(0)e^{-9t/2} + Ce^{-10t}\ . \end{aligned}$$

This approximation suggests that this equilibrium point is a **sink**: regardless of the value of C (which could be determined explicitly if one knew $X(0) = x(0) - 10$ explicitly), all the terms in these expressions approach 0 exponentially fast as $t \to +\infty$. Interpreted in terms of population, this is the equilibrium at which y becomes extinct and x approaches the environmental constraint $x = 10$.

Although we did not consider it in class, let us briefly examine the situation at the point $(100, -9/2)$. (This point does not fit into the predator-prey-modeling scheme since a negative population has no satisfactory interpretation.) The linearized equations become

$$\begin{aligned} \frac{dX}{dt} &= \frac{dx}{dt} = -100X - 2000Y - X^2 - 20YX \approx -100X - 2000Y \\ \frac{dY}{dt} &= \frac{dy}{dt} = -\frac{9}{40}X - \frac{1}{20}XY \approx -\frac{9}{40}X \end{aligned}$$

and the system is reminiscent of the damped linear harmonic oscillator above, but not quite the same: differentiating the second equation and plugging in the first and second gives

$$\begin{aligned} \frac{d^2Y}{dt^2} &= -\frac{9}{40}\frac{dX}{dt} = -\frac{9}{40}\left[-100 \cdot \frac{-40}{9}\frac{dY}{dt} - 2000Y\right] \\ \frac{d^2Y}{dt^2} &+ 100\frac{dY}{dt} - 450Y = 0\ . \end{aligned}$$

It can be shown that the solutions of this equation must have the form $C_1 e^{-\alpha t} + C_2 e^{-\beta t}$, where $\alpha \approx 4.31$ and $\beta \approx -104.31$, and that solutions of the original system then have the form $e^{\alpha t}\mathbf{V}_1 + e^{\beta t}\mathbf{V}_2$, where $\{\mathbf{V}_1, \mathbf{V}_2\}$ is a certain (probably not orthogonal) basis of the XY-plane. Thus this equilibrium point "looks like a source in one direction and like a sink in another," very much like the equilibrium point $(0, 0)$, but the way in which this should be interpreted is not at all clear. (We shall study equilibrium points of this kind [called **saddles**] in detail in the textbook's §5.1, which is the "official" treatment of linearization.)

(ii) The equilibrium points of

$$\begin{aligned} \frac{dx}{dt} &= \frac{3}{10}x - \frac{xy}{100} \\ \frac{dy}{dt} &= 15y\left(1 - \frac{y}{15}\right) + 25xy \end{aligned}$$

are obtained by setting the r. h. sides of these equations equal to zero and solving the simultaneous equations in two unknowns that result; it is routine to find that the solutions are $(0, 0)$ (rather trivially), $(0, 15)$, and $(3/5, 30)$. It is easy to linearize these equations at $(0, 0)$, since that amounts to just considering the terms of degree 0 and 1; we get the linearized system

$$\begin{aligned} \frac{dx}{dt} &= \frac{3}{10}x \\ \frac{dy}{dt} &= 15y \end{aligned}$$

which is again decoupled. The two populations are independent of each other in this approximation, with $x(t) = x(0)e^{3t/10}$ and $y(t) = y(0)e^{15t}$. Note that we may expect this equilibrium point to behave (approximately) like a source for x and y both, provided that $(x(0), y(0))$ is sufficiently close to $(0, 0)$.

If we expand the r. h. sides around the equilibrium point $(0, 15)$ but only keep the linear part, we get

$$\begin{aligned} \frac{dx}{dt} &= \frac{3}{20}x - \frac{1}{100}xY \approx \frac{3}{20}x \\ \frac{dY}{dt} = \frac{dy}{dt} &= 375x - 15Y - Y^2 - 25xY \approx 375x - 15Y \end{aligned}$$

where $Y = y - 15$ (the coördinate centered at the equilibrium point; since the equilibrium value of x is 0, we do not have to transform that coördinate). Because the equation in x does not involve y we can simply solve it: $x' = \frac{3}{20}x$ is a linear homogeneous differential equation whose solution is $x(t) = x(0)e^{3t/20}$. Plugging this expression for $x(t)$ into the equation for Y gives us

$$\begin{aligned}
\frac{dY}{dt} &= 375x(0)e^{3t/20} - 15Y \\
\frac{d}{dt}\left[e^{15t}Y(t)\right] &= 375 \cdot x(0)e^{303t/20} \\
e^{15t}Y(t) &= \frac{20}{303}x(0)e^{303t/20} + C \\
Y(t) &= \frac{20}{303}x(0)e^{3t/20} + Ce^{-15t} .
\end{aligned}$$

This approximation suggests that unless $x(0) = 0$ exactly, both populations will grow from starts near the equilibrium point. Finally, if we set $X = x - 3/5$ and $Y = y - 30$ near the remaining equilibrium point, we get

$$\begin{aligned}
\frac{dX}{dt} = \frac{dx}{dt} &= -\frac{3}{500}Y - \frac{1}{100}XY \approx -\frac{3}{500}Y \\
\frac{dY}{dt} = \frac{dy}{dt} &= 750X - 30Y + 25XY - Y^2 \approx 750X - 30Y
\end{aligned}$$

which gives us

$$\begin{aligned}
\frac{d^2X}{dt^2} = -\frac{3}{500}\frac{dY}{dt} &= -\frac{3}{500}\left\{750X - 30 \cdot \left[\frac{-500}{3}\frac{dX}{dt}\right]\right\} \\
\frac{d^2X}{dt^2} + 30\frac{dX}{dt} + \frac{9}{2}X &= 0 .
\end{aligned}$$

The solutions of this equation have the form $X(t) = C_1e^{\alpha t} + C_2e^{\beta t}$ where $\alpha \approx -0.151$ and $\beta \approx -29.85$, so both $X(t)$ and $Y(t)$ tend to 0 as $t \to \infty$; this suggests that this equilibrium point of the original system is a **sink**.