THINGS TO REVIEW FOR THE FINAL EXAM

Just as with the second hour exam, there is less material here than there seems to be, so try not to panic. You now know much more about the early material of the course than you did when you first saw it, and that will help considerably.

Chapter I contains some material that we haven't looked at since the first hour exam, and it will require review, so the outline of this chapter below will be more detailed than that for later chapters. You may want briefly to look over the Sontag notes "Introduction to First-Order Differential Equations"; I have some extra hard copies and I believe the notes are available on the web. Because so much of the later material of the course concerns only linear equations, §**1.2: Separation of Variables**, one of the few general approaches to nonlinear equations, needs to be looked at. Pick a couple of odd-numbered problems to make sure you haven't lost the technique. Can you still recognize **autonomous equations** and the "pure integration problems" of the form $y' = f(t)$ (where there is no y-dependence on the r. h. s.)? See if you can set up a sample mixing problem like the one done in detail on pp. 30–31.

§**1.3: Slope Fields** are actually a special case of *direction fields*, which are recently familiar: the reason is that the slope fields for the scalar equation $\frac{dy}{dt} = f(t, y)$ are very hard to tell from the direction fields for the system

$$\frac{dx}{dt} = 1$$
$$\frac{dy}{dt} = f(x, y)$$

since the first equation makes $x = t$ (with the initial condition $x(t_0) = 0$). However, you will want to be able to recognize autonomous equations from their slope fields (parallel slopes along *horizontal* lines, a result of invariance under time shift) and similarly for equations $y' = f(t)$ (the general solutions have the form $y(t) = \int_{t_0}^{t} f(\tau)\, d\tau + C$, so $C = y(t_0)$ and changing the initial condition moves the graph vertically; thus these equations have parallel slopes along *vertical* lines). In the case of autonomous equations $y' = f(y)$, you should again be recognizing the occurence of constant solutions $y(t) \equiv y_0$ occurring at **equilibrium values** of y—solutions of the equation $f(y) = 0$. (§1.5 will be devoted to their classification.) Try a few of the odd-numbered problems from §1.3 (we did #11 to death on the first hour exam), but do not get hung up on the particular electrical-engineering applications.[1] Note that you can reverse-engineer such problems as #1–#6: if you find the general solution of the equation, the slope fields become easy.

I do not plan to include §**1.4: Euler's Method** in the final. Take Math 373 and learn what an absolutely rotten numerical approach Euler's method is (and learn some better numerical approximation schemes). For the time being, be a consumer—via `Maple` or `Matlab`—rather than a producer of numerical approximations.

[1] Engineering students should be in Math 244 anyhow.

Things to carry out of §**1.6: Existence and Uniqueness** are: if the r. h. s. function of $y' = f(t, y)$ is continuous, solutions exist *locally*; if $f(t, y)$ is continuously differentiable, the local solution near $t = t_0$ is *uniquely determined by requiring that* $y(t_0) = y_0$. Such equations as $y' = 1 + y^2$ exhibit the fact that solutions—in this case $y(t) = \tan(t - \varphi)$—may "blow up in finite time"; such equations as $y' = 3y^{2/3}$ exhibit the fact that more than one solution satisfying $y(0) = 0$ may exist when the r. h. s. fails to have continuous (or even bounded) partial derivatives. The most useful application of these theorems in this course has been: when uniqueness holds, the graphs of solutions cannot cross—which leads to comparison of solutions, see p. 69 ff.—and in particular, equilibrium solutions of autonomous equations provide "natural barriers" to their growth and decay. This material was well received when we did it, and it should come back to you quickly.

The **equilibria** and **phase line** of §**1.6** should hold no fear for people who have recently been looking at the same phenomena in a 2-dimensional phase space! *Sources* and *sinks* have similar characterizations[2] (positive derivative on the r. h. s. vs. positive eigenvalues for sources; negative for sinks); *nodes* occur when both $f(y) = 0$ and $f'(y) = 0$. Consider the problems in the first hour exam on this subject when you review this material. *Use linearization!* which prepares you for linearization in Chapter V.

§**1.7: Bifurcation** went ok on the first hour exam and should not be a disaster on the final.[3] You should review the drawing of bifurcation diagrams: if the differential equation is $y' = f(y; \mu)$ and μ is the bifurcation parameter, then sketch the graph of the equation $f(y; \mu) = 0$ with μ as the horizontal, y as the vertical coördinate. Vertical tangents give bifurcation values of μ and the regions into which the graph divides the (μ, y)-plane give regions of constant sign for the phase line (see, *e.g.*, Fig. 1.82 on p. 97). If bifurcation occurs at (y_0, μ_0) then both equations $f(y; \mu) = 0$ and $\dfrac{\partial y}{\partial \mu} = 0$ must be satisfied at (y_0, μ_0): this gives an "algebraic" approach to finding *possible* bifurcation. Remember, though, that just as zeros of the derivative sometimes don't give maxima, those points (y_0, μ_0) sometimes don't give bifurcation. Sontag has notes entitled "Some Comments on Bifurcation" on the web; I also circulated a hard-copy version. You may find this to be useful reading.

§**1.8: Linear Differential Equations** is one of the most important things you can take out of this course: we used the same approach for higher-dimensional linear systems too. The textbook authors are longer-winded on this than they needed to be. The basic steps are:

(1) Be sure it's linear. If it contains the unknown function y and/or its derivative in a nonlinear function, or as a power higher than 1, or contains a product $y' \cdot y$, it is not a linear equation.

(2) Get the equation into standard form. It should look like one of

(a)
$$\frac{dy}{dt} = g(t)y + r(t)$$

[2] Centers don't exist in one dimension.

[3] Note that the critical-damping problem on the second hour exam was actually a 2-dimensional bifurcation problem, and most people handled it with little trouble.

$$\textbf{(b)} \qquad \frac{dy}{dt} + a(t)y = r(t)$$

and if you know what you're doing it doesn't matter which, although all the terms involving y and y' should eventually wind up on the l. h. s. Therefore, if it came to you in form **(a)**, get it into form **(b)**, which may introduce a minus sign that wasn't there before. Assuming it's in form **(b)**,

(3) Multiply by $\exp(\int^t a(\tau)\, d\tau)$. The effect will be to make the equation into one in which there is an obvious derivative-of-a-product on the l. h. s. and a function of only t on the r. h. s.:

$$\frac{d}{dt}\left[e^{\int^t a(\tau)\,d\tau} \cdot y\right] = e^{\int^t a(\tau)\,d\tau} y' + a(t)e^{\int^t a(\tau)\,d\tau} = e^{\int^t a(\tau)\,d\tau} r(t)\ .$$

Now take antiderivatives = indefinite integrals of both sides: on the r. h. side you get an additive constant. Then solve for $y(t)$. You will have the same **complete success** exhibited on p. 111. Try the odd-numbered exercises of §1.8. That's it for Chapter I.[4]

Chapter II contains mostly-qualitative material, and since `Maple` is not available for examinations, little can be done with it. My feeling was that it offered a place to begin looking at linearization at equilibria, so you have notes that linearize a few of the §§2.1–2.2 problems. If you look at those notes again you will see how much easier things are now that we know about first-order linear systems. For example, the LIN SYS on p. 2 of those notes has the matrix $\begin{pmatrix} 0 & -4/3 \\ 3/2 & 0 \end{pmatrix}$ with zero trace and positive determinant = 2, so its trajectories are ellipses centered at the equilibrium point—no special pleading needed. The linearized system on p. 3 has matrix $\begin{pmatrix} -10/9 & -4/3 \\ 2/3 & 0 \end{pmatrix}$ and its complex eigenvalues and negative trace show it spiraling in to the equilibrium point, which is a (spiral) sink. This material is systematically treated in §§5.1–5.2, and reading those §§ in conjunction with these and the notes should give you adequate insight into what you can find out without computer technology. The "analytic methods for special systems" of §2.3 have essentially been superseded by the matrix methods we now know. Decoupled systems—linear ones, at least—are recognized as upper- or lower-triangular matrix systems, and we have done the (sometimes damped) harmonic oscillator of §2.3 to death.

So what should you be able to do? Find equilibrium points, as exemplified in the early §2.1 problems or §2.2 problems 11–16. Linearize at equilibrium points, as in the notes or §5.1 problems, and be able to do source/sink/saddle classifications after linearization (you know how to do this: the systems are linear again!). Geometrically, given a direction-field sketch near an equilibrium point you should be able to recognize sources/sinks/saddles (even if nonlinear: could you predict the form of the linearized equations at the equilibrium points (0,1) and (0,0) in the figure at the top of p. 172?). §2.3 has been completely

[4] Phew! But there will be much less to say for succeeding chapters, which are closer to recent experience.

superseded by developments in later chapters.[5] §2.4 we omitted because if you think Euler's method is hideously inaccurate in one variable, you've never tried it in two. §2.5 is a 3-dimensional nonlinear example; we've been very cautious in even studying 3-dimensional homogeneous linear systems, and we omitted this altogether.

Chapter III is the center of gravity of the course. Fortunately, most everyone seems to have this material under control. The fact that a first-semester linear-algebra course is prerequisite for Math 251 has meant that you will for the most part be able to skim briskly through §3.1, saying "uh-huh" at appropriate points. You probably will not want to do many problems. In §3.2 you recognize the authors' "straight-line solutions" as eigenvalue/eigenvector solutions: if the system is $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ and $\mathbf{V}$ is a (nonzero) eigenvector of $\mathbf{A}$ belonging to the eigenvalue λ, then $\mathbf{Y}(t) = ke^{\lambda t}\mathbf{V}$ is a solution of the system. If the eigenvalue/vector involve complex numbers there are some complications, but if they are real it is clear what is going on. In particular, if there are enough real eigenvectors/values of $\mathbf{A}$ to span the space (which means two for these 2-dimensional systems) then the general solution of the system is given by

$$\mathbf{Y}(t) = k_1 e^{\lambda_1 t}\mathbf{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2 \ .$$

The reason is: this has value $k_1\mathbf{V}_1 + k_2\mathbf{V}_2$ at time $t = 0$, and because the eigenvectors form a basis of $\mathbb{R}^2$, suitable k_j's can be found to make this value equal any preassigned $\mathbf{Y}_0$. You should be able to carry out the details of finding the λ_j's, $\mathbf{V}_j$'s and k_j's given a concrete system $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ and initial-position vector $\mathbf{Y}_0$. At this point you may also want to start reading Sontag's "Introduction to Matrix Exponentials" notes, skipping over the considerations involved when the eigenvalues are non-real. Try some of the (non-technological parts of the) §3.2 problems to see that your skills are adequate.

§3.3 introduces the sinks/saddles/sources classification, which seems (from the second hour-exam results) to be fairly well understood. I didn't emphasize the rôle of the **separatrix** of a saddle sufficiently in class, and you may want to read the Paul-and-Bob example on pp. 259–261 carefully to get a hand on what unstable equilibria can do. Most of the problems require phase-plane sketches. I would save my problem time to review §3.4, which deals with (pairs of conjugate) complex eigenvalues. I passed out notes on the purely-real approach to this subject because in past semesters I have encountered enormous resistance to the use of complex exponentials. That resistance did not show up this semester, and that is rewarding. Forming $e^{\lambda t}\mathbf{V}$, even when λ is complex, and then separating the result into real and imaginary parts[6] to get two linearly independent solutions of $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ is efficient and easy: but Euler's relation

$$e^{(\alpha+\beta i)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

[5] This is an example of an unfortunate development in pedagogy and textbook writing: "circling in" on what you really want to do by setting up make-work problems that can be done with the wrong tools, and then doing them the right way later, sometimes much later. Wouldn't it be less tedious simply to learn and use the Y2K tools immediately instead of messing around with what was available in AD 1700?

[6] Of course, the coefficients in $\mathbf{A}$ have to be real in order for this to work.

must be under your control in order for the complex methods to work. Sources and sinks become spirals in this context, and **centers** occur for the first time. One should remember that if one knows the eigenvalues are complex, then (in the 2-dimensional case only!) the **trace** of $\mathbf{A}$ distinguishes between spiral sources and spiral sinks (and centers when present). The **direction of spiraling** (or circling about a center) can be determined by seeing which way the first column of $\mathbf{A}$ points at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (since it's the velocity vector of the trajectory at that point) and/or which way the second column of $\mathbf{A}$ points at $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (same reason).

The odd-numbered problems of §3.3 would probably repay your consideration. After you do one or two in detail, though, you may want to simply use easily computable criteria (trace, determinant, directions that the columns point) to give a qualitative description of the trajectories rather than computing stuff out in full detail (a time-consuming project).

What to do about the **repeated eigenvalues** of §3.4? (Zero eigenvalues don't represent anything really new.) The good 2-dimensional approach is probably that of Sontag's matrix-exponential notes. However, the approach of the textbook on pp. 285–286 is really the perspicuous one that generalizes to higher dimensions, so it is the one I would recommend. In the case of the phase-plane system associated with a second-order linear equation

$$y'' + py' + qy = 0$$

that just happens to have equal eigenvalues (the **critically-damped** case), my inclination is to say: simply remember that the general solution has the form

$$y(t) = k_1\, e^{\lambda t} + k_2\, te^{\lambda t}$$

where λ is the (single, real) eigenvalue, and don't worry too much about how it is derived. Real insight into these questions—particularly in higher dimensions—comes from the *Jordan canonical form* for matrices, and that in turn[7] comes from the principal part of the Laurent series of $(\lambda\mathbf{I} - \mathbf{A})^{-1}$ at the pole represented by the eigenvalue—whatever that all means. So: try a few of the odd-numbered problems of §3.5, using the authors' method. You might want to peek at the answer to insure that you are looking at a genuine doubled-eigenvalue problem, and avoid problems like #1, which is predigested (the matrix is already [lower-]triangular, so exponentiating it *à la* the Sontag notes would be extremely easy).

For the **second-order linear equations** of §**3.6** I unhesitatingly recommend the "method of the lucky guess." Do **not** go to a phase-plane approach unless it is specifically requested: not only does that take longer and give no more information, but it encourages

[7] For an analyst, like your humble servant—algebraists have a completely different and perverted view of this subject.

nonsense[8] "solutions" as answers to exam questions. Oscillators we understand: try maybe one of the odd-numbered problems of this § to make sure you can carry out the details of fitting a general solution to specific initial conditions (try to do one in which neither $y(0) = 0$ nor $y'(0) = 0$, so that you have to face a nontrivial system of equations to determine the undetermined coefficients).

§**3.7** was not on the syllabus. We did not spend enough time on the **3-dimensional linear systems** of §**3.8**, which is unfortunate. The skew-symmetric case is important, and I have prepared (obviously *optional*) notes on it which will be on the web (or I can give interested parties a hard copy). The only case of these that we really know how to handle is the case in which the 3×3 matrix in $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ has 3 real eigenvalues/vectors, and for that the solutions are just the functions

$$\mathbf{Y}(t) = k_1 e^{\lambda_1 t}\mathbf{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2 + k_3 e^{\lambda_3 t}\mathbf{V}_3$$

that a sensible person would expect.

We are very much up to recent history now, and I don't want to beat to death material of which most people have recently demonstrated mastery. To review **Chapter IV**, therefore, I suggest that you look at the recent review notes for the second hour exam, look at your second hour exam, see what (if anything) you had trouble with, and fix it. In particular, do not go to the phase plane to do second-order linear equations. If they are homogeneous, use the "lucky guess" method. If they are being driven by a forcing function that is a polynomial in t, an exponential, or a sinusoidal function (= circular function of ωt for some $\omega \in \mathbb{R}$), use an appropriate lucky-guess or undetermined-coefficient method, as described in §4.1 and (for sinusoidal forcing) §4.2. Be able to find steady-state solutions and their amplitudes (I am less concerned with phase—you can't hear phase except when one of your speakers is hooked up backward).

Chapter V is getting unfortunately short shrift. Thank goodness we looked at linearization in Chapter II. For §§5.1–5.2, the recommendation is that you try finding equilibria and linearizing on a few of the odd-numbered problems #7–17 of §5.1 and a few nonlinear nullclines from odd-numbered problems #5–14 of §5.2. I definitely want to look at §5.3, **Hamiltonian systems** today, last day of class or not, because they are where the conservation laws of physics come from and because they offer substantial insight into a number of things (for example, a first-order system that is Hamiltonian is a trace-zero matrix, and in the complex-eigenvalue case the ellipses that are its trajectories come right out of the conservation law—er, I mean, the Hamiltonian function). These are easy and fun.

[8] For example, vector-valued solutions to scalar equations.