THINGS TO REVIEW FOR EXAM 2

The material to be covered is most of the textbook's Chapter 3: §§ 3.1–3.6, and §1 of Chapter 4. There is less material here than there seems to be, so try not to panic.

Most of §1 is a review of linear algebra and matrix manipulation that will already be familiar from your first-semester linear-algebra course, and of that only the 2×2 case occurs. The only DE material that is covered is **(a)** the process of turning a higher-order linear scalar DE into a linear system of first-order DEs: this is first introduced on p. 212, but you have seen it repeatedly since then; and **(b)** the principle of linearity, or "doing things term-by-term," which is beaten to death on pp. 221-227. You may want briefly to notice how familiar the considerations of problems 16, 17 and 19 on pp. 230–231 seem to you now.

Some substance begins to appear in §2, in which "straight-line solutions" are introduced. Of course you now know that these are solutions of $\mathbf{Y}' = \mathbf{AY}$—here $\mathbf{A}$ is a constant matrix and $\mathbf{Y}(t)$ is a vector-valued function—that have the form $\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}$ where λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{V}$ is an eigenvector belonging to λ. Much of §3.2 is devoted to a review of those fundamental algebraic concepts: if you don't have them under control already, this is the place to work on them. Remember that if a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is known to have a nontrivial null space—for example, if it is $\lambda\mathbf{I} - \mathbf{A}$ where λ is known to be an eigenvalue of $\mathbf{A}$—then $\pm \begin{pmatrix} b \\ -a \end{pmatrix}$ will automatically belong to the null space. (If this happens to be the zero vector, use the bottom row instead of the top row!) In the case in which $\mathbf{A}$ has two distinct real eigenvalues, one can already compute general solutions for $\mathbf{Y}' = \mathbf{AY}$ at this point: pick a couple of odd-numbered problems from among 1–14 on pp. 247–249 and do them for practice. Make sure that when you can find the general solution of $\mathbf{Y}' = \mathbf{AY}$ you can also find the particular solution satisfying $\mathbf{Y}(0) = \mathbf{Y}_0$ where $\mathbf{Y}_0$ is given concretely, "with numbers;" this process usually involves solving a 2×2 system of linear equations. If the system has been produced from a higher-order scalar equation, do not disparage the "guess-and-test" method to find the general solution of the higher-order equation directly: plug e^{st} into one of the equations in problems 21–25, p. 250 (these are really Ch. 2 problems), and determine s for which the exponential is a solution. (The equation you have to solve for s will be the same as the characteristic equation you would have if you converted the higher-order equation into a first-order system.) This is discussed further in Ch. 4.

§3.3 discusses in great detail the case in which $\mathbf{A}$ has two distinct real eigenvalues. The crucial source/sink/saddle classification is introduced here (pp. 251–258); most people had that sorted out on Quiz 4, and it is survival information on this examination. These equilibria are **stable** when they are sinks and **unstable** in other cases (p. 259). While it is a bit much to expect you to give good free-hand sketches of direction fields and trajectories (aka solution curves) for these equations, you can reasonably be expected to recognize them from `Maple`-produced sketches. You should also be able to attack such problems as 17 and

19 on p. 262 by finding the eigenvalues/vectors and general solutions, and seeing what effect the choice of initial points has on the long-term behavior of unstable systems.

§3.4 handles the case of two complex eigenvalues, necessarily each other's conjugates when $\mathbf{A}$ is real. In class we spent quite some time looking at this case from the standpoint of real coördinates, and you have notes for this: that approach generalizes to higher dimensions. For computational purposes with 2×2 systems, however, the method of finding real and imaginary parts of complex eigenvectors—pp. 267–269—should be preferred. Note that if $\lambda = \alpha + \beta i$ is a complex eigenvalue of $\mathbf{A}$, then most of the **qualitative** information you might need about the solutions can be obtained from the matrix itself. The real part α of the complex eigenvalues is $\mathrm{tr}(\mathbf{A})/2$: the origin is a **(spiral) sink** if the trace is negative and a **(spiral) source** if it is positive. If $\mathrm{tr}(\mathbf{A}) = 0$, then the origin is a **center** (and the trajectories are closed curves, namely ellipses or circles). The angular velocity of the solutions is given by β, so their period is $2\pi/\beta$. The direction of travel is most easily determined by remembering that the velocity vector of a point moving along a trajectory is $\mathbf{AY}$ when the point is at position $\mathbf{Y}$. Since

$$\text{if } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } \mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \text{ and } \mathbf{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix},$$

the columns of $\mathbf{A}$ give the velocity vectors at the tips of the standard basis vectors; these "little arrows" make it easy to perceive the direction of rotation. If you can handle the odd-numbered problems 1–15 and 23 of this section (with perhaps just one rather impressionistic sketch of a trajectory!) you will have no problems with the exam questions on this material.

We soft-pedaled the material of §3.5 in class because the cases it considers are to some extent idealizations. Repeated negative eigenvalues still give sinks—and stability; repeated positive ones still give sources. A system with one zero eigenvalue is intrinsically unstable: a slight change in $\mathbf{A}$ can push that eigenvalue positive or negative without changing the sign of the other eigenvalue, so these can turn from sinks to saddles or from sources to saddles with a small change of data in applications. The most important case (and this is easier to see in §3.6) occurs for the simple-harmonic-oscillator equation $y'' + py' + qy = 0$: $q > 0$ is a "spring constant" and somewhat intrinsic to a mechanical system, but if the "damping coefficient" or "friction coefficient" p can be controlled, the choice $p = 2\sqrt{q}$ gives what's called **critical damping**: if the damping coefficient is exactly $2\sqrt{q}$ the system will come to approximate rest without oscillating, but even if p is a little smaller the oscillations will be very slow and will damp out very quickly (this material is discussed fully in §3.6, pp. 307–308). I do not plan to ask detailed questions about the §3.5 material: the important cases are handled in §3.6.

The §3.6 material occurs in all sorts of applications, mostly because Newton's $F = ma$ is a second-order differential equation; so you should understand this material thoroughly. Fortunately, you can view it qualitatively from the two-dimensional-system standpoint, and thus you know what to expect. You should find general solutions for these equations by the "lucky-guess" method, because you understand where it comes from; the only peculiarity[1] is that in the case of a repeated root λ of $s^2+ps+q=0$ the general solution is $k_1e^{\lambda t}+k_2te^{\lambda t}$. You should be able to do the odd-numbered problems on p. 309 without difficulty, but don't devote all your study time to doing all of them! it's only an 80-minute exam.

The material of §§3.7–3.8 is not covered in this examination.

It is very nearly the case that if you understand the §3.6 material you will automatically understand the material of §§4.1–4.2. The following two paragraphs summarize all you will need to understand of the latter.

The crucial points are: linearity of the "left-hand side operator" $y \mapsto y''+py'+qy$ implies that if $k_1e^{\lambda_1 t}+k_2e^{\lambda_2 t}$ (or its circular-function analogues when the characteristic roots are complex) is the general solution of $y''+py'+qy=0$ and y_{part} is *some* solution of $y''+py'+qy=g(t)$, then $k_1e^{\lambda_1 t}+k_2e^{\lambda_2 t}+y_{\text{part}}$ is the general solution of $y''+py'+qy=g(t)$. Thus the problem of finding the general solution of an inhomogeneous equation is reduced to the problem of finding y_{part}, combined with a problem that we have already completely solved. Moreover, linearity implies that if the r. h. side $g(t)$ is a sum of several terms, then y_{part} can be found by working term-by-term.

For most "driving functions" $g(t)=e^{st}$, a suitable choice of C will produce a particular solution Ce^{st}; usually $C=1/(s^2+ps+q)$. The exception occurs when s is an eigenvalue of the homogeneous equation (and thus a root of the denominator in that choice of C); in that case, try te^{st}: see **Second Guessing** on pp. 354 ff. If the driving function is a polynomial of degree k, try a polynomial of the same degree with undetermined coefficients—unless zero is a root of the characteristic equation, in which case try a polynomial of degree $k+1$ (or $k+\ell$ if zero is a root of multiplicity ℓ). If the driving function is a polynomial of degree k times e^{st} , try a polynomial of the same degree with undetermined coefficients times e^{st}—unless s is a root of the characteristic equation, in which case try a polynomial of degree $k+1$ (or $k+\ell$ if s is a root of multiplicity ℓ). The coefficients are determined by stuffing that polynomial into the l. h. side of the equation, simplifying (grouping terms of the same degree in t), and comparing coefficients on the l. h. and r. h. sides (a "missing" term is really present, but its coefficient is zero). If $\cos\omega t$ or $\sin\omega t$ occurs in the driving function, replace it with $e^{i\omega t}$ and proceed as with e^{st}; then take the real part of the solution if $\cos\omega t$ occurred in the driving function, but take the imaginary part if the driver involved $\sin\omega t$. You will not be expected to regurgitate the material contained in the notes on variation of parameters, nor to produce anything resembling accurate phase-plane graphs in this situation (as pp. 369–371 unintentionally demonstrate, even a machine can't do that very well). If you can do the odd-numbered problems 1–15 of §4.2 (pp. 370–371), your mastery of this material is adequate at this stage.

[1] Of course, because you know how to exponentiate a matrix with a repeated eigenvalue but only one eigenvector, you know where these solutions come from.