MATH 252:01 — HOUR EXAM 2

Suggested Solutions

1. Let $\mathbf{A} = \begin{pmatrix} -6 & 5 \\ -4 & 3 \end{pmatrix}$ in the following questions **(a)**, **(b)**, and **(c)**.

(a) Find the eigenvalues of $\mathbf{A}$ and **find an eigenvector** belonging to each eigenvalue.

We have $(\lambda I - A) = \begin{pmatrix} \lambda+6 & -5 \\ 4 & \lambda-3 \end{pmatrix}$, *so* $\det(\lambda I - A) = (\lambda+6)(\lambda-3)+20 = \lambda^2+3\lambda+2 = (\lambda+1)(\lambda+2) = 0$, *with roots* $\lambda = -1$ *and* $\lambda = -2$. *Putting* $\lambda = -2$ *gives* $\lambda I - A = \begin{pmatrix} 4 & -5 \\ 4 & -5 \end{pmatrix}$ *with null solutions spanned by* $\mathbf{V}_1 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$. *Putting* $\lambda = -1$ *gives* $\lambda I - A = \begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix}$ *with null solutions spanned by* $\mathbf{V}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) Find the general solution of the system $\dfrac{d\mathbf{Y}(t)}{dt} = \mathbf{AY}(t)$, using your results of **(a)**.

The general solution is given by taking linear combinations of the "straight-line solutions" associated with the eigenvalue/eigenvector pairs, i.e., $\mathbf{Y}(t) = k_1 e^{-2t}\begin{pmatrix} 5 \\ 4 \end{pmatrix} + k_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(c) Using your results of **(b)**, find the solution of $\mathbf{Y}' = \mathbf{AY}$ for which $\mathbf{Y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Since **(b)** *gives* $\mathbf{Y}(0) = \begin{pmatrix} k_1 \cdot 5 + k_2 \\ 4 \cdot k_1 + k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, *we have to solve the linear system* $5k_1 + k_2 = 1$, $4k_1 + k_2 = 0$. *The solution is* $k_1 = 1$, $k_2 = -4$ *so the solution* $\mathbf{Y}(t) = 1 \cdot e^{-2t}\begin{pmatrix} 5 \\ 4 \end{pmatrix} - 4 \cdot e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5e^{-2t} - 4e^{-t} \\ 4e^{-2t} - 4e^{-t} \end{pmatrix}$ *is the solution satisfying the given initial conditions.*

2. Let $\mathbf{A} = \begin{pmatrix} -1 & -8 \\ 2 & -1 \end{pmatrix}$ in the following questions **(a)**, **(b)**, and **(c)**.

(a) **Find** the eigenvalues of $\mathbf{A}$ (which will be complex).

Here $\det(\lambda I - A) = \det\begin{pmatrix} \lambda+1 & 8 \\ -2 & \lambda+1 \end{pmatrix} = (\lambda+1)^2+16 = 0$, *so* $\lambda = -1 \pm 4i$. *(Expanding the polynomial into* $\lambda^2 + 2\lambda + 17 = 0$ *and using the quadratic formula would give the same result.)*

(b) **Find** the (complex) eigenvector belonging to one (your choice) of the eigenvalues you found in **(a)**.

With $\lambda = -1 \pm 4i$ *one has* $\lambda I - A = \begin{pmatrix} \pm 4i & 8 \\ -2 & \pm 4i \end{pmatrix}$; *the null solutions of this matrix are spanned by* $\mathbf{V}_\pm = \begin{pmatrix} \pm 2i \\ 1 \end{pmatrix}$. *Note that any complex scalar multiple of one of these eigenvectors will also be an eigenvector belonging to the same eigenvalue, so correct results for this and for* **(c)** *below may look considerably different from the solutions given here.*

(c) Give the elements of the basis of $\mathbb{R}^2$ that is involved in solving the differential equation $\mathbf{Y}' = \mathbf{AY}$ using the eigenvector you found in **(a)**.

The real and imaginary parts of $\mathbf{V}_\pm$ *are* $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *and* $\begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}$ *respectively. Again, a different choice of the complex eigenvector can lead to apparently different results here that are no less correct.*

3. The eigenvalues of $\mathbf{A} = \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix}$ are $\lambda = \pm 3i$, and a complex eigenvector belonging to $3i$ is $\begin{pmatrix} 4+3i \\ 5 \end{pmatrix}$. Use that information to answer the following questions **(a)**, **(b)**, and **(c)**.

(a) What is the complex solution of $\mathbf{Y}' = \mathbf{AY}$ associated with the complex eigenvalue and eigenvector given above?

$$e^{3it}\begin{pmatrix} 4+3i \\ 5 \end{pmatrix} = (\cos 3t + i \sin 3t)\begin{pmatrix} 4+3i \\ 5 \end{pmatrix} = \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ 5\cos 3t \end{pmatrix} + i\begin{pmatrix} 4\sin 3t + 3\cos 3t \\ 5\sin 3t \end{pmatrix}.$$

(b) Using the complex solution you gave in **(a)** above, **give two linearly independent $\mathbb{R}^2$-valued solutions** of $\mathbf{Y}' = \mathbf{AY}$.

Here one wants the real and imaginary parts of the complex solution found above,

$$\mathbf{Y}_{\text{real}}(t) = \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ 5\cos 3t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_{\text{imag}}(t) = \begin{pmatrix} 4\sin 3t + 3\cos 3t \\ 5\sin 3t \end{pmatrix}.$$

(c) State whether the origin is a source, a sink or a center for the system $\mathbf{Y}' = \mathbf{AY}$.

Since the real part of the complex eigenvalue is zero, the origin is a center for the system; the trajectories are closed curves (ellipses).

4. (a) The homogeneous second-order linear differential equation

$$(*) \qquad \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

governs the motion of an overdamped spring. **Find the general solution** of $(*)$.

Trying $y(t) = e^{st}$ *leads to the equation* $s^2 + 3s + 2 = (s+1)(s+2) = 0$ *as a necessary and sufficient condition for this function to satisfy* $(*)$, *so any function of the form* $y(t) = k_1e^{-t} + k_2e^{-2t}$ *satisfies the equation, and since the two terms are linearly independent, this is the general solution.*

(b) Let $C > 0$ be a positive real number. **Find the particular solution** of $(*)$ that satisfies $y(0) = 1$, $y'(0) = -C$. Your solution will contain the number "C" in one or more places.

For the general solution found for **(a)** *above, we have* $y(0) = k_1 + k_2$ *and* $y'(0) = -k_1 - 2k_2$, *so the coefficients have to satisfy* $k_1 + k_2 = 1$ *and* $-k_1 - 2k_2 = -C$. *Solving gives* $k_2 = C - 1$ *and* $k_2 = 2 - C$, *so the solution of* $(*)$ *satisfying the given initial conditions is* $y(t) = (2-C)e^{-t} + (C-1)e^{-2t}$.

(c) Explain, by giving an equation satisfied by t_1, why your solution of **(b)** shows that if $y'(0) = -C$ is sufficiently large and *negative*, there will exist a time $t_1 > 0$ for which $y(t_1) = 0$. (This is plausible from the mechanical standpoint: if the mass is displaced positively from equilibrium and thrown *back, hard enough, toward the equilibrium position*, it will pass through the equilibrium position once [in fact, exactly once].)

Setting the function $y(t)$ *obtained in* **(b)** *equal to zero gives the equation* $y(t) = (2-C)e^{-t} + (C-1)e^{-2t} = 0$ *as the condition that our "*t_1*" must satisfy. This equation is equivalent to* $(2-C)e^t + (C-1) = 0$ *or* $e^t = \dfrac{C-1}{C-2}$. *This has a solution with* $t > 0$ *if and only if the r. h. s. is* > 1, *and the inequality* $\dfrac{C-1}{C-2} > 1$ *holds for* $C > 0$ *if and only if* $C > 2$. *So an initial velocity* > 2 *directed toward the equilibrium is needed to make the moving mass cross the equilibrium position. (It can do that only once, since the function* $(2-C)e^t + (C-1)$ *is a decreasing function of* t; *the unique solution is our* t_1.)

5. Find a particular solution of the (inhomogeneous) second-order linear DE

$$(**) \qquad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = -10\,\cos t\ .$$

You may use either a complex-exponential "good guess" method or a method of undetermined coefficients (substitute $y = A\cos\omega t + B\sin\omega t$ for a suitably chosen ω, compare coefficients on the r. h. and l. h. sides of $(**)$, then solve the resulting equations for the constants A and B).

In the "good guess" method one replaces the driving function $-10\cos t$ by the complex exponential $-10e^{it}$—whose real part is $-10\cos t$—and tries to determine a constant C for which $y(t) = Ce^{it}$ solves $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = -10e^{it}$. The result of the substitution is $(-1-i-2)e^{it} = Ce^{it}$, which is equivalent to $C = \frac{-10}{-3-i} = 3-i$. Then $y(t) = (3-i)e^{it} = (3-i)(\cos t + i\sin t) = (3\cos t + \sin t) + i(3\sin t - \cos t)$, so the real part $y_{\text{part}}(t) = 3\cos t + \sin t$ of this function is a particular solution of the original DE $(**)$.[1]

With the method of undetermined coefficients, one seeks a solution of the form $y(t) = A\cos t + B\sin t$ (the choice $\omega = 1$ is due to the fact that the original driving function was $\cos\omega t$ with $\omega = 1$). Successive differentiation, and multiplication by the coefficients of the equation, yields

$$\begin{aligned} -2\cdot \quad y &= A\cos t + B\sin t \\ -1\cdot \quad y' &= B\cos t - A\sin t \\ 1\cdot \quad y'' &= -A\cos t + -B\sin t \\ y'' - y' - 2y &= (-3A - B)\cos t + (A - 3B)\sin t = -10\cos t \end{aligned}$$

which upon comparison of the coefficients of $\cos t$ and $\sin t$ on both sides of the last equal sign gives the equations $-3A - B = -10$ and $A - 3B = 0$. Solving these equations simultaneously gives $B = 1$ and $A = 3$, so $y(t) = 3\cos t + \sin t$, the same particular solution as that obtained by the "good guess" method.[2]

6. (a) Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} + y = 0\ .$$

Trying $y(t) = e^{st}$ leads to the equation $s^2 + 2s + 1 = 0$, as in the preceding problem. Since $s^2 + 2s + 1 = (s+1)^2 = 0$ has $s = -1$ as a root of multiplicity 2, there is only one solution of the form e^{-t}, but te^{-t} is a second solution linearly independent of e^{-t}. The general solution has the form $y(t) = k_1e^{-t} + k_2te^{-t}$.[3]

(b) Solve the initial-value problem

$$\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} + y = 2\,\cos 2t$$
$$y(0) = 0, \quad y'(0) = 0\ .$$

The first step is to find a particular solution of this DE. Again the "good guess" method using complex exponentials is most efficient, so we try for a solution of the equation $\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} + y = 2e^{2it}$, of which we shall then take the real part. If the solution is to have the form Ce^{2it}, then substituting this into the equation and solving for C gives

$$C = \frac{2}{(2i)^2 + 2\cdot 2i + 1} = \frac{2}{-3+4i} = \frac{-6-8i}{25}$$

(1) For a sample application of this technique, see the BD&H example on p. 367 (before the page turn—they're only looking for a particular solution at this point).

(2) For an example of this technique (though it is not carried out in detail) see BD&H's problem 15, p. 371.

(3) For a discussion of this "critically damped oscillator" situation showing this second independent solution, see BD&H, p. 307.

and so $\dfrac{-6-8i}{25}\cdot(\cos 2t + i\sin 2t)$ *satisfies the equation* $y''+2y'+y=2e^{2it}$. *It follows that the real part of this function, which is* $\dfrac{-6\cos 2t+8\sin 2t}{25}$, *is a particular solution of the original differential equation.*

In view of the results of **(a)**, the general solution[4] of the (inhomogeneous) equation $y''+2y'+y=2\cos 2t$ is $y(t)=k_1e^{-t}+k_2te^{-t}+\dfrac{-6\cos 2t+8\sin 2t}{25}$. We now have to determine the coefficients k_1 and k_2 to make $y(0)=0=y'(0)$. Evaluation, and differentiation followed by evaluation, give the initial position and derivative of this function as

$$y(0)=k_1+\frac{-6}{25}$$
$$y'(0)=-k_1+k_2+\frac{16}{25}$$

and setting these equal to zero and solving gives $k_1=\dfrac{6}{25}$, $k_2=\dfrac{-10}{25}$. The final result is thus

$$y(t)=\frac{6}{25}\,e^{-t}-\frac{10}{25}\,te^{-t}+\frac{-6\cos 2t+8\sin 2t}{25}\,.$$

7. Below are four first-order linear homogeneous 2×2 systems of ordinary differential equations. **Label each equation with the answers to the following questions** about the equilibrium at $\mathbf{0}$. You need not give details, but no partial credit will be given.

(a) Are the eigenvalues of the matrix of the system real or complex-and-not-real?

(b) If the eigenvalues are real, is $\mathbf{0}$ a source, a sink or a saddle?

(c) If the eigenvalues are nonreal, is $\mathbf{0}$ a source, a sink or a center, and do the solutions move in a counter-clockwise (positive) or clockwise (negative) angular direction?

(i)

$$\frac{d\mathbf{Y}}{dt}=\begin{pmatrix}1 & -5\\ 2 & -4\end{pmatrix}\mathbf{Y}(t)$$

The characteristic equation is $\lambda^2+3\lambda+6=0$, *with complex eigenvalues* $\lambda=\dfrac{-3\pm i\sqrt{15}}{2}$. *This is a spiral sink and the columns of the matrix indicate that the trajectories spiral counterclockwise (positively).*

(ii)

$$\frac{d\mathbf{Y}}{dt}=\begin{pmatrix}9 & -10\\ 8 & -9\end{pmatrix}\mathbf{Y}(t)$$

The characteristic equation is is $\lambda^2-1=0$ *with real roots* $\lambda=\pm 1$; *the origin is a saddle point.*

(iii)

$$\frac{d\mathbf{Y}}{dt}=\begin{pmatrix}2 & 2\\ 1 & 3\end{pmatrix}\mathbf{Y}(t)$$

The characteristic equation is $\lambda^2-5\lambda+4=0$ *with roots* $\lambda=1$ *and* $\lambda=4$, *so the origin is a (real) source.*

[4] A similar problem, with this method of finding the general solution of an inhomogeneous second-order equation, is given in detail on BD&H's pp. 363–364.

(iv)

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 9 & 2 \\ -30 & -6 \end{pmatrix} \mathbf{Y}(t)$$

The characteristic equation is $\lambda^2 - 3\lambda + 6 = 0$ *with roots (by the quadratic formula)* $\lambda = \dfrac{3 \pm i\sqrt{15}}{2}$*; it is a spiral source and the columns of the matrix indicate that the trajectories spiral in the negative (clockwise) direction.*