Math 252:01 — Spring 2002
MTh3 SEC-212
Prof. Bumby

Project 2: A famous autonomous system

Project should be handed in by Thursday, April 11. Grade will be based on content as well as on clarity and neatness of presentation. The use of a computer is neither required nor prohibited.

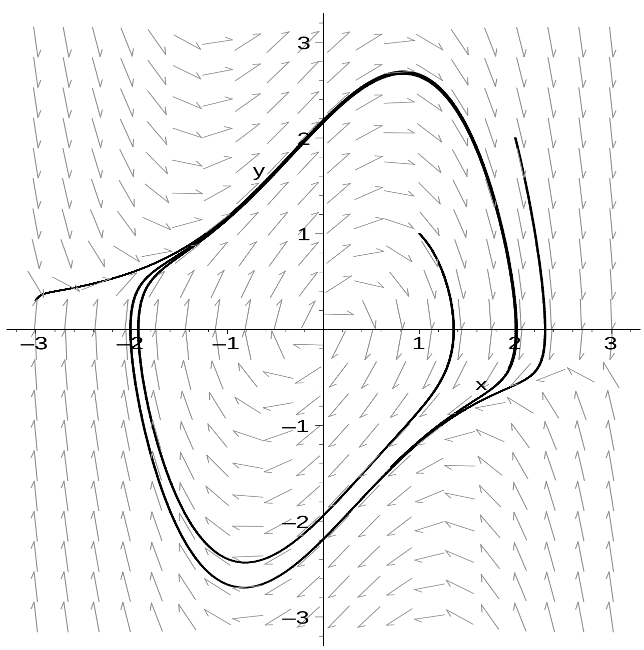
1. Introduction. The system

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x + (1 - x^2)y\end{aligned} \qquad (V)$$

is described in Section 2.4 (page 187) of the text. It is equivalent to the second-order equation

$$\frac{d^2x}{dt^2} - (1 - x^2)\frac{dx}{dt} + x = 0,$$

which is known as the *Van der Pol equation*. The equation arose from a model of certain vacuum-tube circuits, and was subsequently found to model biological processes. The *observed behavior* of the circuit included a stable oscillation. Important theoretical work arose from seeking a proof that the model showed the same behavior. This project follows some of that analysis. Since the independent variable t does not appear on the right side of (V), the system is *autonomous*, allowing the solutions to be studied in a *phase plane* with coordinates x and y. Solutions are characterized, except for a translation in t by their projections, called *trajectories*, into the phase plane. For convenience, here is a graph of the slope field and some solutions.



Some important curves in the phase plane. The condition $dx/dt = 0$ determines the points where the tangent to the solution is perpendicular to the x axis, and the condition $dy/dt = 0$ determines the points where the tangent to the solution is perpendicular to the y axis. Each of these conditions defines a curve that is easily found from (V). The equilibrium points, where $\frac{dy}{dt} = \frac{dx}{dt} = 0$, are clearly on both of these curves.

Exercise 1. Define the **window** W by $-3 \le x \le 3$ and $-3 \le y \le 3$. Graph the two curves described above in the window W. Although the tangent **line** to the trajectories is known where it crosses one of these curves, the direction of increasing t on that line was suppressed in this construction. However, that information is easily recovered. Moreover, the direction of crossing can reverse only at an equilibrium point. Illustrate this by drawing **one** representative of the direction field on each arc of these curves between equilibrium points. Do not show a computer-generated direction field in this graph — only the points where the field is parallel to an axis are of interest here. You can draw the graphs with a computer, but you should add the directions by hand.

Equilibrium points and linearization. Since dx/dt is only zero on the line $y = 0$ (the x axis), and substituting $x = 0$ into the expression for dy/dt gives $dy/dt = -x$ on the x axis, the only equilibrium point is $(0, 0)$.

Since products of small numbers (i.e., numbers close to zero) are *very* small, near the origin the expressions for the derivatives in (V) are dominated by the terms of degree 1. The system containing just the linear terms

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x + y \end{aligned} \tag{L}$$

is called the **linearization** of the system (V) at $(0, 0)$. A discussion of this example appears in Section 5.1 of the textbook, which we will get to soon enough.

Exercise 2. Show that (L) has a **spiral source** at the origin (see Section 3.4).

Angular Motion. The angle θ from the positive x axis can be defined as $\arctan(y/x)$ when $x > 0$. Thus

$$\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2 + y^2}. \tag{A}$$

This expression makes sense *everywhere* except for $(x, y) = (0, 0)$, and allows a *running total* of the angle θ to be computed on any curve. For closed curves, the line integral of this expression, once around the curve, is $2n\pi$ for an integer n. Here, n is called the **winding number** of the curve around the origin. For curves enclosing the origin once in the counterclockwise direction, the winding number is $+1$, and reversing the direction of travel replaces the winding number by its negative.

Thus it is interesting to consider a curve where

$$x\frac{dy}{dt} = y\frac{dx}{dt}. \tag{Θ}$$

This curve locates the points where the tangent points directly towards, or directly away from, the origin. On one side of this curve the trajectories move around the origin in a counterclockwise direction; on the other side, the motion is clockwise. Any equilibrium point is clearly on this curve.

Exercise 3. First, show that the origin is an isolated solution of (Θ). That is, no point near $(0,0)$ except for $(0,0)$ itself satisfies equation (Θ). Then find the other solutions of (Θ) inside the window W, make a copy of the graph obtained in Exercise 1 and add this graph to it. Also, as in Exercise 1, add **one** representative of the direction field on each arc of this curve between equilibrium points. (This makes sense even though the origin is the only critical point, and the curve doesn't *pass through* the origin, because the graph of (Θ) has more than one branch.

Exercise 4. Your figure should have symmetry with respect to "reflection in the origin", i.e. if (x, y) belongs to the figure, so does $(-x, -y)$. Show that the solutions to the equation have the same symmetry.

Global behavior of trajectories. The curves obtained in Exercise 3 divide the plane into a number of regions. If you start at a point in one region and trace a solution to (V) in either the positive or negative direction, the solution will either cross one of the bounding curves or escape to infinity. For this example, only the negative direction leads to infinity. In the positive direction, all solutions have $d\theta/dt < 0$ for large t, so all solutions tend towards a clockwise motion.

Measuring time from the trajectories. Since $dx/dt = y$ on any trajectory, it is also true that $dt/dx = 1/y$, so time can be recovered as a line integral

$$\int \frac{dx}{y}$$

along any arc of a trajectory where $y \neq 0$. Similarly,

$$\int \frac{dy}{-x + (1 - x^2)y}$$

measures time along arcs of trajectories where $-x + (1 - x^2)y \neq 0$. Moreover, combining these formulas with expressions for derivatives of other quantities with respect to t allows anything expressible in terms of t, x and y to be written as a line integral. This would not seem to be useful since the trajectories are not known, but the same qualitative analysis that allows us to obtain a rough sketch of the solutions from a direction field allows us to estimate these integrals.

Indeed, Picard's proof of the existence and uniqueness of solutions of initial value problems exploits this idea by showing that evaluating such integrals using good enough approximate solutions will lead to better approximations.

Although these formulas need to be used with care because they have denominators that can be zero, any trajectory can be subdivided into arcs where one or the other of these integrals will be defined.

Moreover, once we have decided to use line integrals, more complicated expressions can be investigated. Consider, for example,

$$\int \frac{y\,dx - x\,dy}{x^2 - xy + y^2 + x^3 y}.$$

A direct calculation from (V) shows that

$$y\frac{dx}{dt} - x\frac{dy}{dt} = x^2 - xy + y^2 + x^3 y$$

on any trajectory, so this is yet another line integral along trajectories the measures time.

Using the usual polar coordinate formulas $x = r\cos\theta$, $y = r\sin\theta$, so that

$$\frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta$$
$$\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$$

Thus, $y(dx/d\theta) - x(dy/d\theta) = -r^2$, so this integral simplifies to

$$\int \frac{-r^2\,d\theta}{r^2 - r^2\cos\theta\sin\theta + r^4\cos^3\theta\sin\theta} = -\int \frac{d\theta}{1-\cos\theta\sin\theta + r^2\cos^3\theta\sin\theta} \approx -\int \frac{d\theta}{1-\cos\theta\sin\theta}$$

if r is small. Note that the negative sign signifies that t decreases as θ moves in the positive (counterclockwise) direction. Since the integrand in the approximate integral is periodic, the integral will take the same value P on any interval of length 2π. This value P is the approximate time for a solution of (V) that is close to the origin to wind once around the origin.

Exercise 5. Compute P. *Maple* knows enough tricks to be able to evaluate this approximate period integral exactly.

Distance and Green's Theorem. Another interesting function to consider is the distance from the origin. The *square* of the distance is more useful because it leads to easier calculus. First: its derivative with respect to t is

$$\begin{aligned}\frac{d}{dt}r^2 &= \frac{d}{dt}(x^2+y^2) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ &= 2x(y) + 2y\big(-x + (1-x^2)y\big) = 2y^2(1-x^2)\end{aligned} \tag{$*$}$$

If we multiply the integrands of any of the integrals along trajectories that give t by this expression, the result will be an integral such that the integral along a trajectory is the difference of the value of r^2 at the endpoints of the trajectory.

If the endpoints are connected by an arc on which the line integral is zero, Green's Theorem

$$\oint A\,dx + B\,dy = \iint B_x - A_y\,dx\,dy.$$

can be applied. Here, the integral on the left follows a closed curve in the counterclockwise direction and the integral on the right is the integral over the region bounded by the curve with respect to area. The subscripts in the integral on the right represent partial derivatives.

Care must be taken when using Green's Theorem since it is only valid when both the line integral and the double integral exist. In particular, points that make the denominator zero must be avoided, not just on the path, but also in the interior of a region on which Green's Theorem is to be applied.

Exercise 6. First, integrate $(*)$ along a trajectory using $dt = dx/y$, and find some arcs on which this integral is zero that can be used to give a closed path. Apply Green's Theorem to find an equivalent double integral. Find a line integral with respect to y that is equivalent to this by Green's Theorem.

Remark. The integral obtained in Exercise 6 can be used to prove that there is a unique closed orbit that other orbits approach. While all the clues have been given, the proof is too technical to be given as an exercise, so we stop with finding the integral.