1. Cauchy-Schwarz inequality: $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$ triangle inequality: $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.
Assume Cauchy-Schwarz. Then

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 \le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 = (\|\mathbf{u}\| + \|\mathbf{v}\|)^2.$$

2. (a) $\|\mathbf{u}\| = \sqrt{1+9+4} = \sqrt{14}$, $\|\mathbf{v}\| = \sqrt{4+1+9} = \sqrt{14}$, $\mathbf{u}\cdot\mathbf{v} = -2+3-6 = -5$,
$\|\mathbf{u}+\mathbf{v}\| = \sqrt{1+16+1} = \sqrt{18}$.

(b) Cauchy-Schwarz inequality for these vectors: $|\mathbf{u}\cdot\mathbf{v}| = 5 \le \|\mathbf{u}\|\|\mathbf{v}\| = 14$
triangle inequality for these vectors: $\|\mathbf{u}+\mathbf{v}\| = \sqrt{18} \le \|\mathbf{u}\| + \|\mathbf{v}\| = 2\sqrt{14}$

(c) $(\mathbf{u}+2\mathbf{w})\cdot(\mathbf{u}-\mathbf{w}) = \mathbf{u}\cdot\mathbf{u} + (-1+2)\mathbf{u}\cdot\mathbf{w} - 2\mathbf{w}\cdot\mathbf{w} = 14 + 13 - 2(25) = -23$.

3. (a) Calculate $\mathbf{v}\cdot\mathbf{x} = 1+0-3 = -2$ and $\mathbf{v}\cdot\mathbf{v} = 1+4+9 = 14$. Then $\mathbf{y} = \dfrac{\mathbf{v}\cdot\mathbf{x}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = -\dfrac{1}{7}\begin{bmatrix}1\\2\\3\end{bmatrix}$. Hence

$\mathbf{z} = \begin{bmatrix}1\\0\\-1\end{bmatrix} + \dfrac{1}{7}\begin{bmatrix}1\\2\\3\end{bmatrix} = \dfrac{1}{7}\begin{bmatrix}8\\2\\-4\end{bmatrix}$. Check: $\mathbf{v}\cdot\mathbf{z} = (1/7)(1\cdot 8 + 2\cdot 2 - 3\cdot 4) = 0$.

(b) The vector $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ is in $V^\perp$ when $x_1 + 2x_2 + 3\mathbf{x}_3 = 0$. So $V^\perp = \mathrm{Null}(A)$, where $A = \mathbf{v}^T$. Since x_2 and x_3 are the free variables, a basis for the null space is $\mathbf{u}_1 = \begin{bmatrix}-2\\1\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}-3\\0\\1\end{bmatrix}$.

(c) Apply Gram-Schmidt to the vectors $\mathbf{u}_1$, $\mathbf{u}_2$ from (b):

$\mathbf{v}_1 = \mathbf{u}_1$ and $\mathbf{v}_2 = \mathbf{u}_2 - \dfrac{\mathbf{v}_1\cdot\mathbf{u}_2}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 = \mathbf{u}_2 - \dfrac{6}{5}\mathbf{v}_1 = \dfrac{1}{5}\begin{bmatrix}-3\\-6\\5\end{bmatrix}$. Check: $\mathbf{v}_1\cdot\mathbf{v}_2 = (1/5)((-2)\cdot(-3)+1\cdot(-6)+0\cdot 5) = 0$,
$\mathbf{v}\cdot\mathbf{v}_1 = 1\cdot(-2) + 2\cdot 1 + 3\cdot 0 = 0$, and $\mathbf{v}\cdot\mathbf{v}_2 = (1/5)(1\cdot(-3) + 2\cdot(-6) + 3\cdot 5) = 0$.

For an orthonormal set use $\mathbf{q}_1 = \dfrac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \dfrac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\\0\end{bmatrix}$ and $\mathbf{q}_2 = \dfrac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \dfrac{1}{\sqrt{70}}\begin{bmatrix}-3\\-6\\5\end{bmatrix}$.

(d) The general formula is $c_1 = \mathbf{x}\cdot\mathbf{q}_1 = \dfrac{-2}{\sqrt{5}}$ and $c_2 = \mathbf{x}\cdot\mathbf{q}_2 = \dfrac{-8}{\sqrt{70}}$ since $\|\mathbf{q}_1\| = \|\mathbf{q}_2\| = 1$.

Check: $c_1\mathbf{q}_1 + c_2\mathbf{q}_2 = \dfrac{-2}{5}\begin{bmatrix}-2\\1\\0\end{bmatrix} + \dfrac{-8}{70}\begin{bmatrix}-3\\-6\\5\end{bmatrix} = \dfrac{1}{70}\begin{bmatrix}80\\20\\-40\end{bmatrix} = \mathbf{z}$.

4. Let $R = \mathrm{rref}(A) = \begin{bmatrix}1&1&1&0&0\\0&0&0&1&1\end{bmatrix}$, basic variables x_1, x_4 and free variables x_2. x_3, x_5.

(a) $\dim \mathrm{Row}\, A = \dim \mathrm{Col}\, A = 2$, $\dim \mathrm{Null}\, A = 3$.

(b) For $\mathrm{Col}\, A = \mathbb{R}^2$, use any orthonormal basis, such as $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

For $\mathrm{Row}\, A = \mathrm{Row}\, R$, use $[1\ \ 1\ \ 1\ \ 0\ \ 0]/\sqrt{3}$ and $[0\ \ 0\ \ 0\ \ 1\ \ 1]/\sqrt{2}$.
For $\mathrm{Null}\, A$, the equations are $x_1 + x_2 + x_3 = 0$ and $x_4 + x_5 = 0$, so it has a basis

$\mathbf{u}_1 = \begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}-1\\0\\1\\0\\0\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}0\\0\\0\\-1\\1\end{bmatrix}$. Here $\mathbf{u}_1 \perp \mathbf{u}_3$ and $\mathbf{u}_2 \perp \mathbf{u}_3$. Replace $\mathbf{u}_2$ by

$\mathbf{v}_2 = \mathbf{u}_2 - \dfrac{\mathbf{u}_2\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 = \mathbf{u}_2 - \dfrac{1}{2}\mathbf{u}_1 = \begin{bmatrix}-1/2\\-1/2\\1\\0\\0\end{bmatrix}$. To get an orthnomal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, take

$\mathbf{w}_1 = \dfrac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1 = \dfrac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}$, $\mathbf{w}_2 = \dfrac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \sqrt{\dfrac{2}{3}}\begin{bmatrix}-1/2\\-1/2\\1\\0\\0\end{bmatrix}$, $\mathbf{w}_3 = \dfrac{1}{\|\mathbf{u}_3\|}\mathbf{u}_3 = \dfrac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\0\\-1\\1\end{bmatrix}$.

5. The columns of Q must be an orthnormal basis for $\mathbb{R}^3$. One solution is $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$. (There are infinitely many other solutions,)

6. (a) True: There are four distinct eigenvalues, hence four independent eigenvectors.
(b) False: Eigenvalue 0 has algebric multiplicity 2, but only one eigenvector if nullity $A = 1$.
(c) True: Since nullity $A = 4 - 2 = 2$, there is a linearly independent set of eigenvectors.
(d) True: Symmetry guarantees that the algebraic multiplicity = geometric multiplicity for eigenvalues, so there is a basis of eigenvectors. The diagonal matrix could have entries $2, 0, 0, 0$, or $2, 2, 0, 0$, or $2, 2, 2, 0$, depending on the multiplicities.

7. (a) Characteristic polynomial $\det\begin{bmatrix}(7-t) & 0 & 0\\ 0 & (4-t) & 2\\ 0 & 1 & (3-t)\end{bmatrix} = (7-t)(t^2-7t+10) = -(t-2)(t-5)(t-7)$

Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 5$, $\lambda_3 = 7$. Eigenvectors:

$$A - \lambda_1 I_3 = \begin{bmatrix}5 & 0 & 0\\ 0 & 2 & 2\\ 0 & 1 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}, \text{ so eigenvector } \mathbf{v}_1 = \begin{bmatrix}0\\-1\\1\end{bmatrix}. \quad (\text{Check: } A\mathbf{v}_1 = \begin{bmatrix}0\\-2\\2\end{bmatrix} = 2\mathbf{v}_1)$$

$$A - \lambda_2 I_3 = \begin{bmatrix}2 & 0 & 0\\ 0 & -1 & 2\\ 0 & 1 & -2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 0\end{bmatrix}, \text{ so eigenvector } \mathbf{v}_2 = \begin{bmatrix}0\\2\\1\end{bmatrix}. \quad (\text{Check: } A\mathbf{v}_2 = \begin{bmatrix}0\\10\\5\end{bmatrix} = 5\mathbf{v}_2)$$

$$A - \lambda_3 I_3 = \begin{bmatrix}0 & 0 & 0\\ 0 & -3 & 2\\ 0 & 1 & -4\end{bmatrix} \to \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}, \text{ so eigenvector } \mathbf{v}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}. \quad (\text{Check: } A\mathbf{v}_3 = \begin{bmatrix}7\\0\\0\end{bmatrix} = 7\mathbf{v}_3)$$

(b) Take $P = [\mathbf{v}_1\ ,\ \mathbf{v}_2\ ,\ \mathbf{v}_3]$. Then P is invertible since the eigenvectors are independent (the eigenvalues are distinct); alternate argument: $\det P = 1$. Take $D = \begin{bmatrix}2 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 7\end{bmatrix}$ (diagonal matrix of eigenvalues).

8. (a) Take $P = [\mathbf{v}_1\ ,\ \mathbf{v}_2\ ,\ \mathbf{v}_3]$ and $D = \begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$. Then $P^{-1} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 2 & -1 & 1\end{bmatrix}$ and

$$A = PDP^{-1} = \begin{bmatrix}2 & 0 & 0\\ 1 & 1 & 0\\ -5 & 2 & -1\end{bmatrix}$$ (lower triangular, with eigenvalues on the diagonal).

(b) Let $\mathbf{y} = P^{-1}\mathbf{x} = \begin{bmatrix}3\\1\\7\end{bmatrix}$. Then $\mathbf{x} = P\mathbf{y} = 3\mathbf{v}_1 + 1\mathbf{v}_2 + 7\mathbf{v}_3$. Apply A^n to each eigenvector:

$$A^n\mathbf{x} = 3 \cdot 2^n\mathbf{v}_1 + 1 \cdot 1^n\mathbf{v}_2 + 7 \cdot (-1)^n\mathbf{v}_2 \approx 3 \cdot 2^n\mathbf{v}_1 \text{ when } n \text{ is large.}$$

9. Since $A\mathbf{x} = 2\mathbf{x}$ we have $2\mathbf{x} \cdot \mathbf{y} = (A\mathbf{x}) \cdot \mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T(A\mathbf{y}) = \mathbf{x} \cdot A\mathbf{y}$ since $A = A^T$. But $A\mathbf{y} = 3\mathbf{y}$, so we get $2\mathbf{x} \cdot \mathbf{y} = 3\mathbf{x} \cdot \mathbf{y}$. This forces $\mathbf{x} \cdot \mathbf{y} = 0$.

10. (a) Take $\mathbf{v} = \begin{bmatrix}1\\3\\2\end{bmatrix}$ (column #1 of A). Then $A\mathbf{v} = \mathbf{v}(\mathbf{v}^T\mathbf{v}) = 14\mathbf{v}$, so the eigenvalue is 14.

(b) rref $A = \begin{bmatrix}1 & 3 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$, so nullity$(A) = 2$. Hence the zero eigenspace of A has dimension 2. Vectors $\mathbf{x}$ in Null(A) satisfy $x_1 + 3x_2 + 2x_3 = 0$ (free variables x_2 and x_3), so a basis for Null(A) is

$\mathbf{u}_1 = \begin{bmatrix}-3\\1\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}-2\\0\\1\end{bmatrix}$. Here $\mathbf{v} \cdot \mathbf{u}_1 = \mathbf{v} \cdot \mathbf{u}_2 = 0$ because A is symmetric and $\mathbf{v}$ is an eigenvector with nonzero eigenvalue.

(c) Apply Gram-Schmidt to the vectors $\mathbf{u}_1$, $\mathbf{u}_2$ as in problem #3:

Let $\mathbf{v}_1 = \mathbf{u}_1$, $\mathbf{v}_2 = \mathbf{u}_2 - \dfrac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{u}_2 - \dfrac{6}{10}\mathbf{v}_1 = \begin{bmatrix}-1/5\\-3/5\\1\end{bmatrix}$, $\mathbf{v}_3 = \mathbf{v}$.

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis of eigenvectors for A. Set $\mathbf{q}_i = \dfrac{1}{\|\mathbf{v}_i\|}\mathbf{v}_i$ to get an orthonormal basis of eigenvectors.

(d) Let $Q = [\mathbf{q}_1\ ,\ \mathbf{q}_2\ ,\ \mathbf{q}_3]$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 14 \end{bmatrix}$.

11. (a) False in general: $\operatorname{Null} A \perp \operatorname{Row} A$ (true if $A = A^T$). (b) True: $Q^{-1} = Q^T$.
(c) True: $(P^TQ)^T = Q^TP = Q^{-1}P = Q^{-1}(P^T)^{-1} = (P^TQ)^{-1}$.
(d) True: $\operatorname{Null} A \neq 0$, so $\operatorname{Col} A \neq \mathbb{R}^n$.
(e) False in general: $Q^{-1} = Q^T$ (true if $Q = Q^T$). (f) False in general (true if $A = A^T$).

12. (a) $c_j = \mathbf{w}_j \cdot \mathbf{u} = \mathbf{w}_j^T\mathbf{u}$ (product of row vector and column vector).
(b) The vector $\mathbf{y} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k \in W$ and $\mathbf{u} - \mathbf{y} \perp W$. Hence $\mathbf{y} = \mathbf{w}$ by uniqueness.
(c) $C^T\mathbf{u} = \begin{bmatrix} \mathbf{w}_1^T\mathbf{u} \\ \vdots \\ \mathbf{w}_k^T\mathbf{u} \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ \mathbf{c}_k \end{bmatrix}$, so $CC^T\mathbf{u} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k = P_W\mathbf{u}$.
(d) The i, j entry in C^TC is $\mathbf{w}_i \cdot \mathbf{w}_j$, which is 1 if $i = j$ and 0 if $i \neq j$. Hence $C^TC = I_k$. Since $W = \operatorname{Col} C$, the general formula $P_W = C(C^TC)^{-1}C^T$ simplifies to $P_W = CC^T$.

13. (a) For a line $y = a_0 + a_1x$ in the (x, y) plane with y-intercept a_0 and slope a_1 let
$$E = [9 - (a_0 - 3a_1)]^2 + [7 - (a_0 - a_1)]^2 + [5 - a_0]^2 + [1 - (a_0 + 4a_1)]^2$$
This is the sum of the squares of the vertical distance from the given data points to the line. In the method of *least squares* we choose the coefficients a_0 and a_1 to minimize the error E.
(b) The matrix C has first column all 1, and second column the x data values. The vector $\mathbf{y}$ has the y data values. The unknown vector $\mathbf{u}$ has the intercept and slope of the line. Hence
$$C = \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 0 \\ 1 & 4 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 9 \\ 7 \\ 5 \\ 1 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}.$$
(c) Since $C^TC = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 26 \end{bmatrix}$ and $C^T\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 9 \\ 7 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ -30 \end{bmatrix}$,
the equation for $\mathbf{u}$ is $\begin{bmatrix} 4 & 0 \\ 0 & 26 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 22 \\ -30 \end{bmatrix}$. Hence $a_0 = 22/4 = 11/2$ and $a_1 = -30/26 = -15/13$. The equation of the best-fitting line is $y = (11/2) - (15/13)x$.